BUCKLING ANALYSIS OF STRESS-UNILATERAL STRUCTURAL SYSTEMS

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ABSTRACT

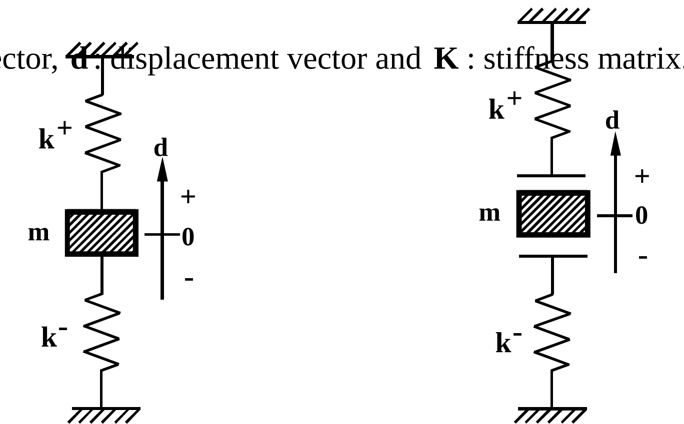
Cable structures, membrane structures, etc. reveal the structural characteristics such that materials used for these structures cannot transmit the compression stress, and then these structures belong to stress-unilateral structural systems. In the paper, the buckling of hybrid structures which consist of rigid bars and cables is treated. These hybrid structures are one kind of stress-unilateral structural systems and have two problems about (1) the construction of consistent stiffness matrix and (2) the selection of higher buckling load in the buckling analysis. The paper describes some illustrative examples of these two problems and presents an analytical method in order to avoid these problems.

INTRODUCTION

Fig.1 shows two kinds of spring-mass models. In the case(a), the spring constant (k) is the same as $k = k^{+} + k^{-}$ in spite of the displacement direction. On the contrary, in the case(b), the spring constant depends upon the direction of displacement such that $k = k^{+}$ for $d > 0$ and $k = k^{-}$ for $d < 0$. The case(a) and the case(b) are called “bilateral stress system” and “unilateral stress system”, respectively [1,2]. For a cable member due to the compression force, $k^{-} = 0$ is usually assumed in the numerical analysis. Fig.2 shows the force-displacement relations for these spring-mass models. Table 1 shows illustrative examples of unilateral stress systems. Since the magnitude of elastic coefficients depends upon the direction of displacement, the stiffness matrix becomes a function of displacement and the force-displacement relation takes the form :

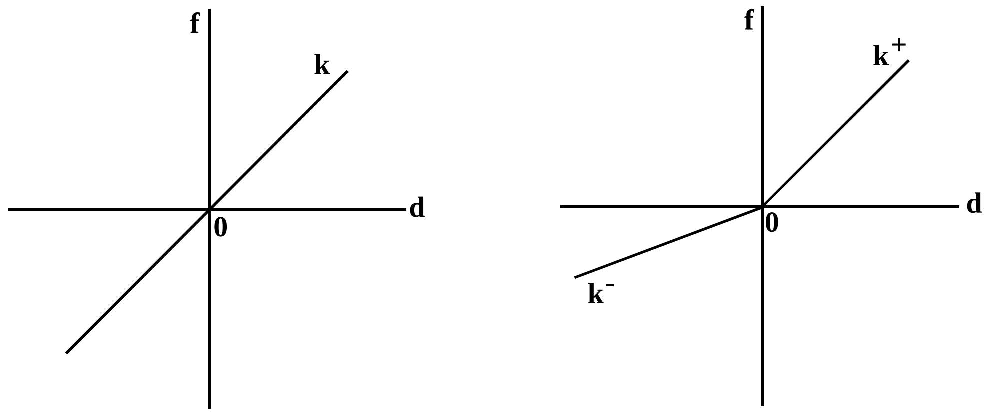
$$\mathbf{f} = \mathbf{K}(\mathbf{d})\,\mathbf{d} \qquad (1)$$

where $\mathbf{f}$: force vector, $\mathbf{d}$: displacement vector and $\mathbf{K}$: stiffness matrix.



(a) Bilateral stress **(b) Unilateral stress**

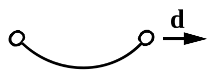
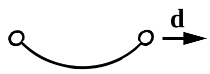
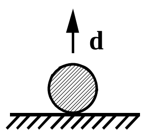
Fig. 1 : Bilateral and Unilateral Spring-Mass Models



(a) Bilateral($k = k^{+} + k^{-}$) **(b) Unilateral**

Fig. 2 : Force-Displacement Relation

Table 1 : Unilateral Stress System

cable	**slack**		$\mathbf{k^{-} = 0}$
membrane	**wrinkle**		$\mathbf{k^{-} = 0}$
contact problem	**constraint of displacement**		$\mathbf{k^{-} = +\infty\ (d^{-}=0)}$

In the beginning of the numerical analysis, **d** is unknown so that the mode of displacement is usually assumed such as

$$\mathbf{d} = \mathbf{d}^{(0)} \tag{2}$$

Introduction of Eq.(2) into Eq.(1) gives us

$$\mathbf{f} = \mathbf{K}(\mathbf{d}^{(0)})\,\mathbf{d} = \mathbf{K}^{(0)}\,\mathbf{d} \tag{3}$$

which gives us

$$\mathbf{d}^{(-1)} = \mathbf{K}^{-1}(\mathbf{d}^{(0)})\,\mathbf{f} \qquad (4)$$

In the similar manner, we have for $r = 0, 1, 2, \cdots\cdots$,

$$\mathbf{f} = \mathbf{K}(\mathbf{d}^{(r)})\,\mathbf{d} = \mathbf{K}^{(r)}\,\mathbf{d} \qquad (5)$$

If $\mathbf{K}^{(r)} = \mathbf{K}^{(r+1)}$ holds, the modes $\mathbf{d}^{(r)}$ and $\mathbf{d}^{(r+1)}$ are coincident. In this case, $\mathbf{K} \equiv \mathbf{K}^{(r)} = \mathbf{K}^{(r+1)}$ is called "consistent stiffness matrix". But, there exist cases where the construction of consistent stiffness matrix is impossible. An example for this case is shown in the following.

Even though we have consistent stiffness matrices, we have other problem about the selection of higher buckling load. If we assume two different initial modes of displacement, say, $\mathbf{d}^{(0)}$ and $\overline{\mathbf{d}}^{(0)}$, then we have two different iteration processes such as

(a) $\mathbf{d}^{(0)} \rightarrow \mathbf{K}^{(0)} \rightarrow \mathbf{d}^{(1)} \rightarrow \mathbf{K}^{(1)} \rightarrow \cdots\cdots$

(b) $\overline{\mathbf{d}}^{(0)} \rightarrow \overline{\mathbf{K}}^{(0)} \rightarrow \overline{\mathbf{d}}^{(1)} \rightarrow \overline{\mathbf{K}}^{(1)} \rightarrow \cdots\cdots$

And, after several iterations, the above processes lead to the consistent stiffness matrices $\mathbf{K}$ and $\overline{\mathbf{K}}$, which give us the buckling loads P_{cr} and $\overline{P}_{cr}$. If these buckling loads are different, say, $P_{cr} > \overline{P}_{cr}$, then we have higher buckling load when we start the iteration process with $\mathbf{d}^{(0)}$. An example about the selection of higher buckling load is also presented in following.

In the last part of the paper, the vibration method is proposed in order to avoid these two problems.

TANGENT STIFFNESS OF UNILATERAL STRESS SYSTEM

As shown in Fig.3, consider node i with two different spring constants : k_i^+ and k_i^-.

If k_i is the spring constant of node i, the relation

$$\begin{aligned} k_i &= k_i^+ \quad \text{for} \quad d_i \geq 0 \\ k_i &= k_i^- \quad \text{for} \quad d_i < 0 \end{aligned} \qquad (6)$$

holds . Eq.(6) gives us

$$k_i \;=\; k_i(d_i) \;=\; \frac{k_i^+}{2}\{1+\mathrm{sgn}(d_i)\} \;+\; \frac{k_i^-}{2}\{1-\mathrm{sgn}(d_i)\} \qquad (7)$$

where

$$\mathrm{sgn}(d_i\;) \;=\; \begin{cases} +1 & \text{for} \quad d_i \geq 0 \\ -1 & \text{for} \quad d_i < 0 \end{cases} \qquad (8)$$

Then, the load-displacement relation of node i takes the form

$$f_i \;=\; \left[\; \frac{k_i^+}{2}\{1+\mathrm{sgn}(d_i)\} \;+\; \frac{k_i^-}{2}\{1-\mathrm{sgn}(d_i)\} \right] d_i \qquad (9)$$

Consider two models : (a) Bilateral spring-rigid bar model(Model-A) and (b) Unilateral spring-rigid bar model(Model-B) as shown in Fig.4 and 5.

The total potential energy ($\prod$) is given by taking the sum of the internal energy(**U**) stored in springs and the potential energy of the external force($-\,\mathbf{P}\cdot\mathit{\Delta}$) as

$$\prod = \mathbf{U} - \mathbf{P}\cdot\mathit{\Delta} \qquad (10)$$

where **P** : the axial force and $\mathit{\Delta}$: the displacement of the end node.

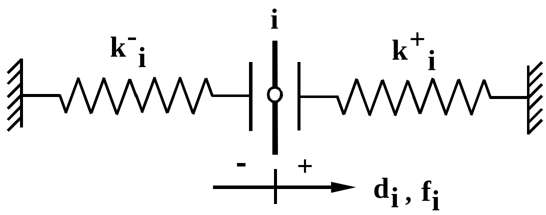


Fig. 3 : Spring of Unilateral Stress System

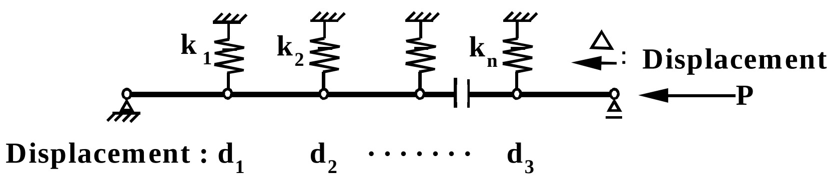


Fig. 4 : Bilateral Spring-Rigid Bar Model(Model-A)

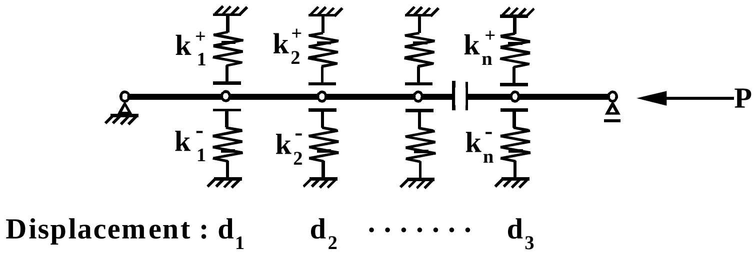


Fig. 5 : Unilateral Spring-Rigid Bar Model(Model-B)

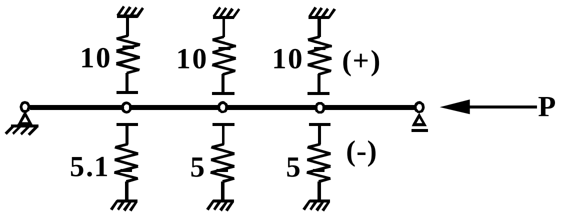


Fig. 6 : Model-1

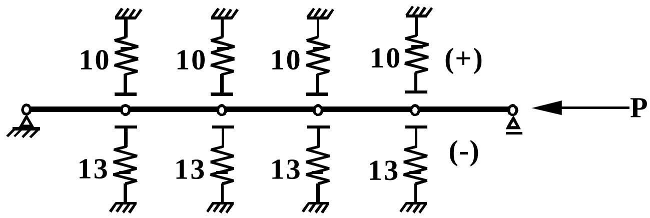


Fig. 7 : Model-2

If we introduce

$$\mathbf{k}_A = \begin{bmatrix} \mathbf{k}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{k}_n \end{bmatrix} \quad , \quad \mathbf{k}_B = \begin{bmatrix} \mathbf{k}_1(\mathbf{d}_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{k}_n(\mathbf{d}_n) \end{bmatrix} \qquad (11)$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & & & \mathbf{0} \\ & \mathbf{2} & & & \\ & & \ddots & & \\ & & & \ddots & \mathbf{-1} \\ \textbf{sym.} & & & & \mathbf{2} \end{bmatrix} , \qquad (12)$$

Eq.(10) takes the form

$$\Pi_A = \frac{1}{2}\mathbf{d^T}\left[\mathbf{k_A} - \frac{P}{l}\mathbf{A}\right]\mathbf{d} = \frac{1}{2}\mathbf{d^T K_A d} \qquad \text{for Model-A} \qquad (13)$$

$$\Pi_B = \frac{1}{2}\mathbf{d^T}\left[\mathbf{k_B}(\mathbf{d}) - \frac{P}{l}\mathbf{A}\right]\mathbf{d} = \frac{1}{2}\mathbf{d^T K_B d} \qquad \text{for Model-B} \qquad (14)$$

Equilibrium equations are obtained by $\frac{\partial \Pi}{\partial d_i} = 0$ as

$$\mathbf{K_A}(P)\mathbf{d} = \mathbf{0} \qquad \text{for Model-A} \qquad (15)$$

$$\mathbf{K_B}(\text{sgn}(\mathbf{d}) \,,\, P)\mathbf{d} = \mathbf{0} \qquad \text{for Model-B} \qquad (16)$$

in which $\mathbf{K_A}(P)$ and $\mathbf{K_B}(\text{sgn}(\mathbf{d}) \,,\, P)$ are the tangent stiffness matrices.

BUCKLING ANALYSIS

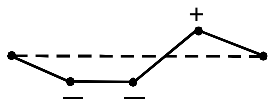
The eigenvalue analysis of Eqs.(15) and (16) gives us buckling loads and buckling modes. At the buckling point,

$$\left| \mathbf{K_A}(P_{cr}) \right| = 0 \qquad \text{for Model-A} \qquad (17)$$

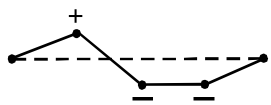
$$\left| \mathbf{K_B}(\text{sgn}(\mathbf{d}) \,,\, P_{cr}) \right| = 0 \qquad \text{for Model-B} \qquad (18)$$

hold. For Model-B, sgn($\mathbf{d}$) is unknown before the buckling mode is determined. Then, an iteration process is usually used. The iteration process is given in Fig.8 where r is the number of iteration step. If the buckling modes $\mathbf{d}^{(r+1)}$ coincidents with $\mathbf{d}^{(r)}$, the tangent matrix $\mathbf{K}^{(r+1)}$ is consistent with $\mathbf{K}^{(r)}$ and the buckling load $P_{cr}^{(r)} = P_{cr}^{(r+1)} = P_{cr}$ is obtained.

But, there exists the case where the construction of consistent stiffness matrix is impossible. Consider Model-1 shown in Fig.6. If we assume the buckling mode (- - +) for $r = 0$, the iteration process becomes as follows.

$r = 0$: − − + 

$r = 1$: + − −

$r = 2$: − − + 

In this case, the iteration process circulates and we are unable to have the consistent tangent stiffness matrix.

Next, an example of the selection of higher buckling load is shown. Fig.7 shows a model named Model-2 and Table 2 shows the iteration process. If we start from the assumed buckling mode ①, the first iteration step gives us the same buckling mode as shown in the first row and we gets the buckling load $P_{cr} = 6.684$. However, if we start from the assumed buckling modes ② ~ ⑩, a few iteration steps give us the same buckling modes as ⑥ and the buckling load $\overline{P}_{cr} = 6.728$ is obtained. $\overline{P}_{cr}$ is higher than P_{cr} so that the different selection of assumed buckling modes give the different buckling loads. These two examples shows the issues about (1) the construction of consistent stiffness matrix and (2) the selection of higher buckling load.

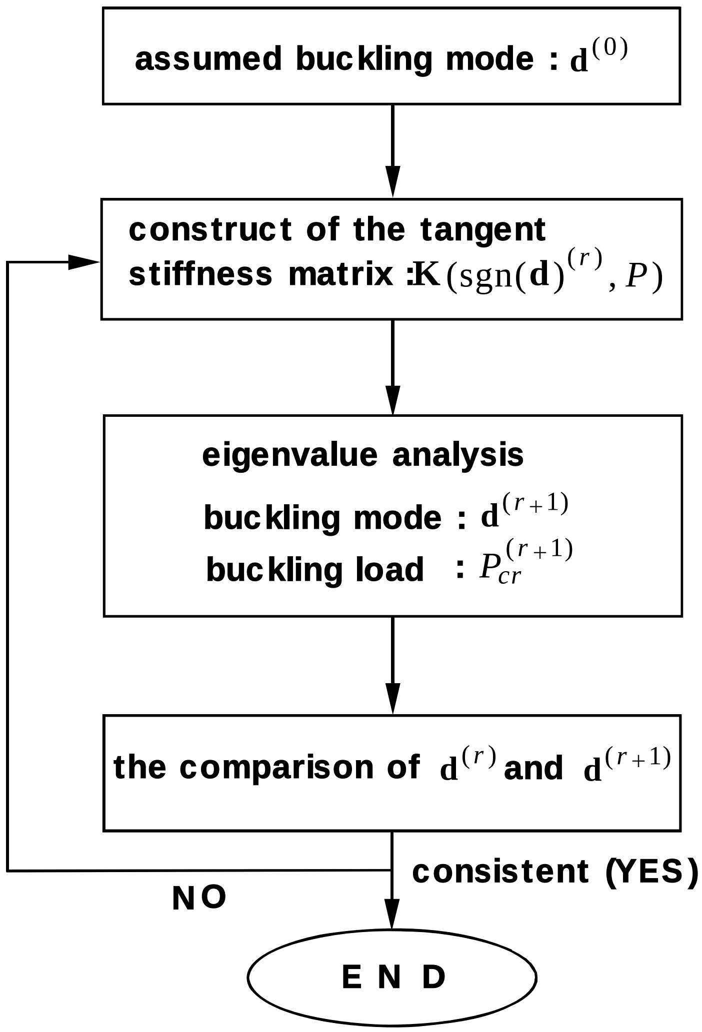


Fig. 8 : Iteration Process

Table 2 : Initial Modes and Buckling Loads

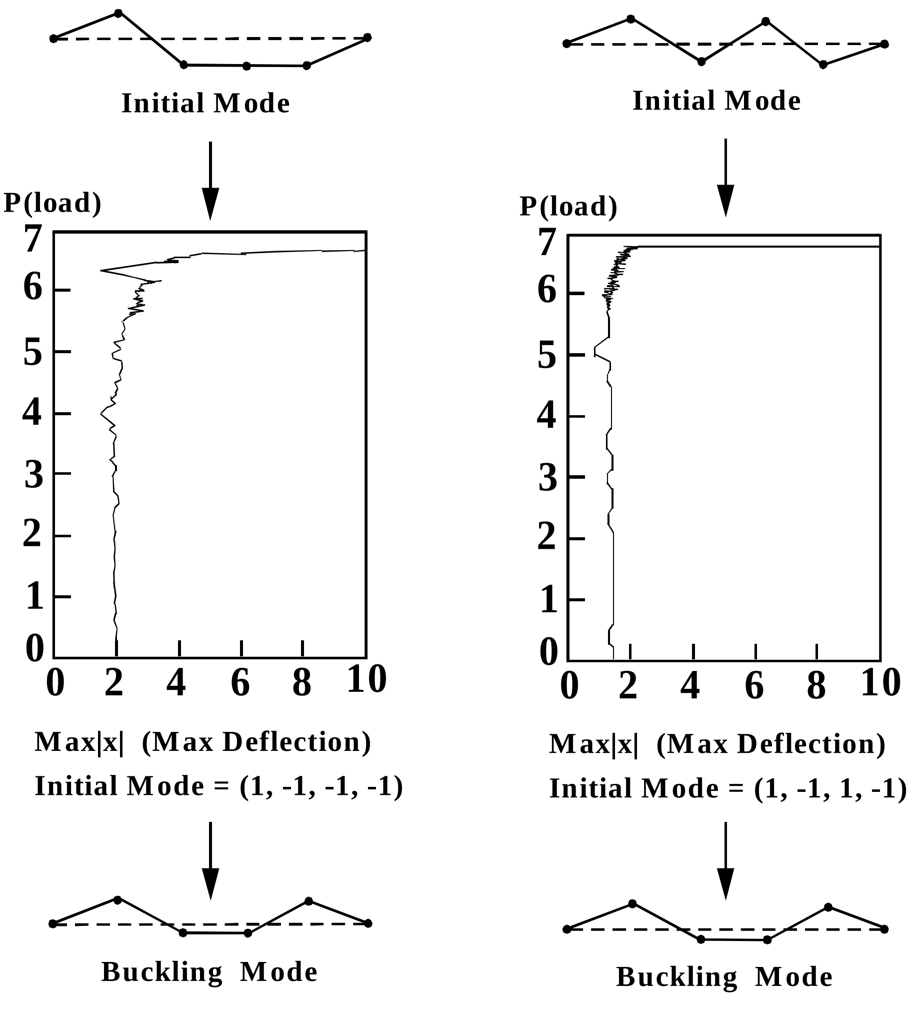
Initial Mode($r=0$)	$r=1$	$r=2$		Buckling Load
①(+ − − +)	①(+ − − +)	−	convergent	$P_{cr}=6.684$
②(+ − − −)	⑤(− + − −)	⑥(+ − + −)	convergent	$\overline{P}_{cr}=6.728$
③(+ + − −)	⑤(− + − −)	⑥(+ − + −)		
④(+ + − +)	⑤(− + − −)	⑥(+ − + −)		
⑤(− + − −)	⑥(+ − + −)	⑥(+ − + −)		
⑥(+ − + −)	⑥(+ − + −)	⑥(+ − + −)		
⑦(+ + + +)	⑥(+ − + −)	⑥(+ − + −)		
⑧(+ + + −)	⑥(+ − + −)	⑥(+ − + −)		
⑨(− + + −)	⑥(+ − + −)	⑥(+ − + −)		
⑩(− − − −)	⑥(+ − + −)	⑥(+ − + −)		

VIBRATION METHOD

To avoid the previous two problems, the vibration method is introduced. The equation of motion for the uni-lateral spring-bar model takes the form :

$$\mathbf{M}\frac{d^2\mathbf{d}}{dt^2} + \mathbf{K}(\mathrm{sgn}(\mathbf{d}),\, P)\mathbf{d} = \mathbf{0} \qquad (19)$$

where **M** is the mass matrix (for example, the case that all nodes have the same unit mass gives $\mathbf{M} = \mathbf{I}$, in which $\mathbf{I}$ is identity matrix) and t denotes time.



(a) case-1 **(b)case-2**

Fig. 9 : Vibration Method

Eq.(19) is a nonlinear equation and is analyzed numerically by Newmark's β method with $\beta = 1/4$. Fig.9 represents the numerical results for Model-2 with two different initial modes. Two graphs show the maximum responses evaluated from each time history curves as the load P increases. The buckling load is defined as the load level at which the maximum displacement response increases suddenly. Fig.9 gives us the buckling load $P_{cr} = 6.684$ and the buckling mode($+ \; - \; - \; +$) which coincide with the buckling load and mode in the first row of Table 2.

The reason that the correct buckling load and mode are obtained by the vibration method can be considered as almost all modes occur during the numerical intergration analysis.

CONCLUSIONS

Two problems about (1) the construction of consistent stiffness matrix and (2) the selection of higher buckling load in the buckling analysis of unilateral structural systems are presented with illustrative examples. To avoid these problems, the vibration method is tried and shown to be an effective method.

REFERENCES

[1] Panagiotopoulos, P.D. : A Variational Inequality Approach to the Inelastic Stress-Unilateral Analysis of Cable-Structures, Computers & Structures, 18, 1984, pp.899-910

[2] Hangai, Y. and Yamagami, T. : Geometrically Nonlinear Vibration of Cable Structures Considering Stress-Unilateral Behaviours, Computational Mechanics ’88, Theory and Application, edited by S.N.Atluri and G.Yagawa, Springer-Verlag, 1988, pp.39ii1-39ii4.