VIBRATION AND BUCKLING OF CIRCULAR CYLINDRICAL SHELL SUBJECTED TO AXIAL STRESSES

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ABSTRACT: Natural frequencies and buckling stresses of thick circular cylindrical shells calculated by using the previously published thin shell theories are usually overpredicted. A two-dimensional higher-order shell theory is applied to the vibration and buckling problems of a simply supported cylindrical shell subjected to axial stresses. Shear deformations and thickness changes have important effects on the natural frequencies and buckling stresses of thick circular cylindrical shells with or without axial stress.

1. INTRODUCTION

A great many significant contributions can be found on vibration and buckling problems of circular cylindrical shells in the literature based upon two-dimensional shell theory. Most of them have been done for thin circular cylinders and very little for thick cylinders. Usually, two approaches have been used to analyze thick shell structures, i.e. one is based on the three-dimensional elasticity theory and the other, approximate two-dimensional shell theory.

It is very complicated to obtain effective solutions of the three-dimensional vibration and buckling problems of thick elastic shells and therefore few papers dealing with such problems of thick circular cylindrical shells have appeared. Based on the three-dimensional theory of elasticity, Armenàkas *et al*. [1] presented a volume containing tables of natural frequencies and graphs of representative mode shapes of harmonic elastic waves propagating in an infinitely long isotropic hollow cylinder. The tables may be used directly in obtaining the frequency of standing waves propagating in simply supported shells of finite length. A finite element method was presented by Bradford and Dong [2] for the vibration and stability analyses of initially stressed orthotropic cylindrical shells. The formulation is capable of treating a three-dimensional initial stress state which is radially symmetric.

In order to take into account the influence of transverse shear deformation and rotatory inertia, a number of authors derived modified shell theories in the past. Mirsky and Herrmann [3] developed a Timoshenko-Mindlin-type shear deformation theory by introducing the shear correction coefficient κ^2, as was done in Timoshenko beams and Mindlin plates. The dynamic shear coefficients were determined by considering the thickness-shear motions in axial and circumferential directions, respectively. By expanding the shell displacement components in power series of the thickness coordinate, there exist approximate two-dimensional shell theories. Upon using certain truncations of the power series, a higher-order shell theory which can take into account the first order effects of transverse shear deformations have been applied to cylindrical shells by Bhimaraddi [4]. Based upon a realistic parabolic variation for shear strains with zero values at the external surfaces, the shear correction factors are not required in the theory. Transverse normal strain is assumed to be zero and transverse normal stress in the direction of the shell thickness is excluded. However, two-dimensional higher-order theories of circular cylindrical shells which take into account the complete effects of shear deformations with thickness changes and rotatory inertia have not been investigated. Recently, it has been pointed out that neglecting higher-order deformations such as shear deformations and thickness changes will lead to an overprediction of the natural frequency and buckling stress for shallow circular arches (Matsunaga [5]) and thick circular rings (Matsunaga [6]).

This paper presents a two-dimensional higher-order theory of thick circular cylindrical shells which can take into account the complete effects of both shear deformations with thickness changes and rotatory inertia. Several sets of the governing equations of truncated approximate theories are applied to the analysis of vibration and buckling problems of a simply supported circular cylindrical shell subjected to axial stresses. Natural frequency and buckling stress for a simply supported circular cylindrical shell subjected to axial stress can be expressed analytically with reference to the corresponding natural frequency for the shell without axial stress. It may be noticed that the two-dimensional higher-order shell theory in the present paper is useful for vibration and buckling problems of very thick circular cylindrical shells.

2. FUNDAMENTAL EQUATIONS OF CIRCULAR CYLINDRICAL SHELLS

Consider a circular cylindrical shell of mean radius of curvature R, thickness H and length L. A curvilinear coordinate system (x, y, z) is defined on the middle surface of the circular cylindrical shell, where the x-axis is taken along the middle surface in the circumferential direction with the y-axis in the axial direction and the

z-axis in the direction normal to the tangent to the middle surface. The dynamic displacement components in a shell are expressed as

$$u = u(x,y,z;t), \qquad v = v(x,y,z;t), \qquad w = w(x,y,z;t) \tag{1}$$

where t denotes time. The displacement components may be expanded into power series of the thickness coordinate z as follows :

$$u = \sum_{n=0}^{\infty} \overset{(n)}{u} z^n, \qquad v = \sum_{n=0}^{\infty} \overset{(n)}{v} z^n, \qquad w = \sum_{n=0}^{\infty} \overset{(n)}{w} z^n. \tag{2}$$

2.1. Strain-displacement relations

Strain components may be expanded as follows :

$$\gamma^{\alpha\beta} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}{}^{\alpha\beta} z^n, \qquad \gamma^{\alpha z} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}{}^{\alpha z} z^n, \qquad \gamma^{zz} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}{}^{zz} z^n \tag{3}$$

where Greek lower case subscripts indicate the coordinate x or y. Strain-displacement relations can be written as (Yokoo and Matsunaga [7])

$$\begin{gathered}
\overset{(n)}{\gamma}_{xx} = \overset{(n)}{u}_{,x} - \frac{1}{R}\overset{(n)}{w} - \frac{1}{R}\left(\overset{(n-1)}{u}_{,x} - \frac{1}{R}\overset{(n-1)}{w}\right), \qquad \overset{(n)}{\gamma}_{yy} = \overset{(n)}{v}_{,y}, \qquad \overset{(n)}{\gamma}_{zz} = (n+1)\overset{(n+1)}{w}, \\
\overset{(n)}{\gamma}_{xy} = \overset{(n)}{\gamma}_{yx} = \frac{1}{2}\left(\overset{(n)}{u}_{,y} + \overset{(n)}{v}_{,x} - \frac{1}{R}\overset{(n-1)}{u}_{,y}\right), \\
\overset{(n)}{\gamma}_{xz} = \frac{1}{2}\left\{(n+1)\overset{(n+1)}{u} - \frac{n-1}{R}\overset{(n)}{u} + \overset{(n)}{w}_{,x}\right\}, \qquad \overset{(n)}{\gamma}_{yz} = \frac{1}{2}\left\{(n+1)\overset{(n+1)}{v} + \overset{(n)}{w}_{,y}\right\}
\end{gathered} \tag{4}$$

where a comma indicates partial differentiation with respect to the coodinate subscripts that follow. No restrictive assumptions are made concerning the order of H/R.

2.2. Equations of motion

Consider a true cylindrical shell subjected to a uniformly distributed initial axial stress s_0 which is assumed to be constant in the axial direction. Since it is assumed that the initial deformation due to the axial stress is axisymmetric and is uniformly distributed in axial direction, there is no influences of the initial deformation in the present problem. Introducing stress components $s^{\alpha\beta}$, $s^{\alpha z}$ and s^{zz}, Hamilton's principle is applied to derive the equations of dynamic equilibrium and natural boundary conditions of a shell. An additional work due to the initial axial stress which is assumed to remain unchanged during vibration and/or buckling is taken into consideration. Both the outer and inner surfaces of a shell are assumed to be traction free. The principle for the present problem may be expressed for an arbitrary time interval t_1 to t_2 as follows :

$$\int_{t_1}^{t_2}\int_V \Big[s^{xx}\delta\gamma_{xx}+s^{yy}\delta\gamma_{yy}+s^{zz}\delta\gamma_{zz}+2(s^{xy}\delta\gamma_{xy}+s^{xz}\delta\gamma_{xz}+s^{yz}\delta\gamma_{yz}) \\ -\rho(\dot{u}\delta\dot{u}_{,x}+\dot{v}\delta\dot{v}_{,y}+\dot{w}\delta\dot{w}_{,z})+s_0(u_{,y}\delta u_{,y}+v_{,y}\delta v_{,y}+w_{,y}\delta w_{,y})\Big]dVdt=0 \tag{5}$$

where the over dot indicates partial differentiation with respect to time and ρ denotes the mass density; dV, the volume element. The volume element is given in terms of normal curvilinear coordinates defined for the middle surface S by

$$dV=\mu\, dzdS, \qquad \mu=1-\frac{z}{R}. \tag{6}$$

The initial axial stress is assumed to be expressed as the following power series :

$$s_0=\sum_{l=0}^{\infty}\overset{(l)}{s_0}\,z^l. \tag{7}$$

By performing the variation as indicated in eqn (5), the equations of motion are obtained as follows :

$$\delta\overset{(n)}{u}:(\overset{(n)}{N}{}^{xx}-\frac{1}{R}\overset{(n+1)}{N}{}^{xx})_{,x}+(\overset{(n)}{N}{}^{xy}-\frac{1}{R}\overset{(n+1)}{N}{}^{xy})_{,y}-n\overset{(n-1)}{Q}{}^{x}+\frac{n-1}{R}\overset{(n)}{Q}{}^{x} \\ +\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\overset{(l)}{s_0}\overset{(m)}{u}_{,yy}\,f(n+m+l+1)=\rho\sum_{m=0}^{\infty}\overset{(m)}{\ddot{u}}\,f(n+m+1) \tag{8}$$

$$\delta\overset{(n)}{v}:\overset{(n)}{N}{}^{xy}{}_{,x}+\overset{(n)}{N}{}^{yy}{}_{,y}-n\overset{(n-1)}{Q}{}^{y}+\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\overset{(l)}{s_0}\overset{(m)}{v}_{,yy}\,f(n+m+l+1)=\rho\sum_{m=0}^{\infty}\overset{(m)}{\ddot{v}}\,f(n+m+1) \tag{9}$$

$$\delta\overset{(n)}{w}:\frac{1}{R}(\overset{(n)}{N}{}^{xx}-\frac{1}{R}\overset{(n+1)}{N}{}^{xx})+\overset{(n)}{Q}{}^{x}{}_{,x}+\overset{(n)}{Q}{}^{y}{}_{,y}-n\overset{(n-1)}{T} \\ +\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\overset{(l)}{s_0}\overset{(m)}{w}_{,yy}\,f(n+m+l+1)=\rho\sum_{m=0}^{\infty}\overset{(m)}{\ddot{w}}\,f(n+m+1). \tag{10}$$

The stress resultants are defined as follows :

$$\overset{(n)}{N}{}^{\alpha\beta}=\int_{-H/2}^{+H/2}\mu s^{\alpha\beta}z^n dz, \qquad \overset{(n)}{Q}{}^{\alpha}=\int_{-H/2}^{+H/2}\mu s^{\alpha z}z^n dz, \qquad \overset{(n)}{T}=\int_{-H/2}^{+H/2}\mu s^{zz}z^n dz. \tag{11}$$

The following functions are defined as

$$f(k)\equiv g(k)-\frac{1}{R}g(k+1) \tag{12}$$

where k is an integer and

$$g(k)\equiv\int_{-H/2}^{+H/2}z^{k-1}dz=\frac{1}{k}\left(\frac{H}{2}\right)^k[1-(-1)^k]=\begin{cases}0 & (k:\text{even})\\ \dfrac{2}{k}\left(\dfrac{H}{2}\right)^k & (k:\ \text{odd}).\end{cases} \tag{13}$$

2.3. Constitutive relations

For elastic and isotropic materials, the constitutive relations can be written as

$$s^{xx} = (2\mu + \lambda)\gamma_{xx} + \lambda(\gamma_{yy} + \gamma_{zz}), \qquad s^{yy} = (2\mu + \lambda)\gamma_{yy} + \lambda(\gamma_{xx} + \gamma_{zz})$$

$$s^{xy} = s^{yx} = 2\mu\gamma_{xy}, \qquad s^{xz} = 2\mu\gamma_{xz}, \qquad s^{yz} = 2\mu\gamma_{yz} \tag{14}$$

$$s^{zz} = (2\mu + \lambda)\gamma_{zz} + \lambda(\gamma_{xx} + \gamma_{yy})$$

where Lamé's constants μ and λ are defined by using Young's modulus E and Poisson's ratio ν as follows :

$$\mu = E / 2(1+\nu), \quad \lambda = \nu E / (1+\nu)(1-2\nu). \tag{15}$$

2.4. Boundary conditions

For the equations of boundary conditions along the boundaries on the middle surface :

$$\overset{(n)}{u} \quad \text{or} \quad \nu_x \overset{(n)}{N}{}^{xx} + \nu_y \overset{(n)}{N}{}^{yx} + \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \nu_y \overset{(l)}{s_0} \overset{(m)}{u}_{,y} f(n+m+l+1)$$

$$\overset{(n)}{v} \quad \text{or} \quad \nu_y \overset{(n)}{N}{}^{yy} + \nu_x \overset{(n)}{N}{}^{xy} + \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \nu_y \overset{(l)}{s_0} \overset{(m)}{v}_{,y} f(n+m+l+1) \tag{16}$$

$$\overset{(n)}{w} \quad \text{or} \quad \nu_x \overset{(n)}{Q}{}^{x} + \nu_y \overset{(n)}{Q}{}^{y} + \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \nu_y \overset{(l)}{s_0} \overset{(m)}{w}_{,y} f(n+m+l+1)$$

are to be prescribed.

2.7. M th order approximate theory

The equations of motion and boundary conditions can be expressed in terms of displacement components. Since the fundamental equations mentioned above are complex, approximate theories of various orders may be considered for the present problem. A set of the following combination of displacement components for Mth ($M \geq 1$) order approximate equations is proposed :

$$u = \sum_{m=0}^{2M-1} \overset{(m)}{u} z^m, \qquad v = \sum_{m=0}^{2M-1} \overset{(m)}{v} z^m, \qquad w = \sum_{m=0}^{2M-2} \overset{(m)}{w} z^m \tag{17}$$

where $m = 0, 1, 2, 3, \cdots, M$.

The total number of the unknown displacement components is $(6M-1)$. In the above cases of $M=1$, an assumption of the normal strain $\gamma_{zz} = 0$ is inherently imposed. Another set of the governing equations of the lowest order approximate theory $(M=1)$ is derived with the use of an assumption that the normal stress s^{zz}

is zero.

Under the assumption of plane state of stresses, the shear strains γ_{xz} and γ_{yz} must vanish through the thickness of a shell and the lowest order approximate theory reduces to the classical shell theory.

3. FOURIER SERIES SOLUTION FOR CIRCULAR CYLINDRICAL SHELL

A simply supported circular cylindrical shell subjected to initial axial stress is analyzed for natural frequencies and vibration modes. Following the Navier solution procedure, displacement components are assumed for the circumferential wave number $r \geq 1$ as

$$\begin{aligned} &\overset{(n)}{u} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{u}_{rs} \cos\frac{rx}{R} \sin\frac{s\pi y}{L} \cdot e^{i\omega t}, \qquad \overset{(n)}{v} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{v}_{rs} \sin\frac{rx}{R} \cos\frac{s\pi y}{R} \cdot e^{i\omega t} \\ &\overset{(n)}{w} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{w}_{rs} \sin\frac{rx}{R} \sin\frac{s\pi y}{R} \cdot e^{i\omega t}, \qquad (0 \leq x \leq 2\pi R, \quad 0 \leq y \leq L) \end{aligned} \tag{18}$$

where the displacement mode number $r = 1, 2, 3, \cdots, \infty$ and $s = 1, 2, 3, \cdots, \infty$, ω denotes the circular frequency and *i*, the imaginary unit. When the circumferential wave number $r = 0$, the following two types of displacement mode may be assumed :

$$\overset{(n)}{u} = \sum_{s=1}^{\infty} \overset{(n)}{u}_{0s} \sin\frac{s\pi y}{L} \cdot e^{i\omega t}, \qquad \overset{(n)}{v} = 0, \qquad \overset{(n)}{w} = 0 \tag{19}$$

$$\overset{(n)}{u} = 0, \qquad \overset{(n)}{v} = \sum_{s=1}^{\infty} \overset{(n)}{v}_{0s} \cos\frac{s\pi y}{L} \cdot e^{i\omega t}, \qquad \overset{(n)}{w} = \sum_{s=1}^{\infty} \overset{(n)}{w}_{0s} \sin\frac{s\pi y}{L} \cdot e^{i\omega t}. \tag{20}$$

Equations (19) and (20) correspond to torsional and axisymmetric vibration modes, respectively.

The equations of motion are rewritten in terms of the generalized displacement components $\overset{(n)}{u}_{rs}$, $\overset{(n)}{v}_{rs}$ and $\overset{(n)}{w}_{rs}$. The present theory yields $(6M-1)$-frequencies for each combination of the displacement mode numbers *r* and *s*. In the following analysis, the axial stress is assumed to distribute uniformly in the thickness direction. Only the first term of the expanded axial stress (7) is considered, *i.e.* $s_0 = \overset{(0)}{s}_0$.

The dimensionless natural frequency and the initial axial stress in the *y*-direction for vibration problems are defined as follows :

$$\Omega = \omega H \sqrt{\frac{\rho}{G}}, \qquad \Lambda = \frac{2\pi R H s_0}{P_c} \tag{21}$$

where G is the shear modulus and P_c is the minimum classical buckling load for the bending problem of a simply supported straight beam of length L with circular cross-section of radius R defined by

$$G = \frac{E}{2(1+\nu)}, \qquad P_c = \frac{\pi^2 EI}{L^2}, \qquad I = \frac{\pi R^4}{4}. \tag{22}$$

4. EIGENVALUE PROBLEM OF A THICK CIRCULAR CYLINDRICAL SHELL

The equations of motion can be rewritten by collecting the coefficients for the generalized displacements of any fixed values r and s. The generalized displacement vector $\{\mathbf{U}\}$ for the Mth order approximate theory is expressed as

$$\{\mathbf{U}\}^T = \{\overset{(0)}{u}_{rs}, \cdots, \overset{(2M-1)}{u}_{rs} \ ; \ \overset{(0)}{v}_{rs}, \cdots, \overset{(2M-1)}{v}_{rs} \ ; \ \overset{(0)}{w}_{rs}, \cdots, \overset{(2M-2)}{w}_{rs} \} . \tag{23}$$

Eigenvalue problems to determine the natural frequency are generalized as follows :

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\{\mathbf{U}\} = 0 \tag{24}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix which contains the effects of initial axial stress and matrix $[\mathbf{M}]$, the mass matrix.

In order to analyze the eigenvalue problems, eqn (24) may be rewritten as follows :

$$([\mathbf{K}]^{-1}[\mathbf{M}] - \frac{1}{\Omega^2}[\mathbf{I}])\{\mathbf{U}\} = 0 \qquad \rightarrow \qquad \det([\mathbf{K}]^{-1}[\mathbf{M}] - \frac{1}{\Omega^2}[\mathbf{I}]) = 0 \tag{25}$$

where matrix $[\mathbf{I}]$ denotes unit matrix.

The matrix $[\mathbf{K}]^{-1}[\mathbf{M}]$ is called the dynamic matrix in the vibration problem. The power method is used to obtain the numerical solution of the eigenvalue problems. Although all the eigenvalues and eigenvectors can be computed by this method for each deformation mode of r and s, the dominant eigenvalues which correspond to the lower natural frequencies are of most concern.

5. NUMERICAL EXAMPLES

5.1. Numerical examples

Natural frequencies of a thick elastic circular cylindrical shell with simply supported boundaries are analyzed in the following numerical examples. When the natural frequency goes to zero under axial compressions, elastic buckling occurs and the buckling stress can be obtained. Since no restrictive assumptions are made concerning the order of thickness-curvature ratio, the upper bound of this parameter is taken to be $H/R = 1.0$. The length parameter L/R is varied from 1 to 20 for short to long circular cylindrical shells. Poisson's ratio is fixed to be $\nu = 0.3$. All the numerical results are shown in the dimensionless quantities.

5.2.Convergence and comparison of natural frequency and buckling stress

In order to verify the accuracy of the present solutions, the convergence properties of the first natural frequency of circular cylindrical shells without axial stress and buckling stress for the displacement mode $r = s = 1$ are shown in Table 1. A direct comparison of the present solutions with those from the classical shell theory (CST) in which the effects of extension and rotatory inertia are included is made. The present results are also compared with those of a first order shear deformation theory (FST) which corresponds to the Mindlin plate theory in which a shear correction factor κ^2 is introduced to correct the contradictory shear stress distribution over the thickness of the shell. The present results for $M = 1-4$ converge accurately enough within the present order of approximate theories.

5.3. Natural frequency and buckling stress

In the case of a simply supported circular cylindrical shell subjected to initial axial stress Λ, the natural frequency Ω_a can be expressed explicitly with reference to the natural frequency Ω_0 of a shell without axial stress.

The relation between Ω_a and Ω_0 can be obtained from a comparison of the equations of motion as follows:

$$\Omega_a{}^2 = \Omega_0{}^2 + \frac{(1+\nu)s^2\pi^4}{4}\left(\frac{R}{H}\right)^3\left(\frac{H}{L}\right)^4\Lambda \tag{26}$$

Table 1. Convergence of solutions and comparison with previously published results ($L/R = 2,\ \nu = 0.3,\ i = j = 1,\ M = 1-5$)

H/R	Ω/Λ	CST	FST	*M*=1	*M*=1 †	*M*=2	*M*=3	*M*=4	*M*=5
0.05	Ω	0.04848	0.04848	0.05014	0.04848	←	←	←	←
	Λ	0.02376	0.02376	0.02541	0.02376	0.02375	←	←	←
0.10	Ω	0.09739	0.09740	0.1008	0.09741	0.09736	←	←	←
	Λ	0.04793	0.04796	0.05134	0.04796	0.04791	←	←	←
0.20	Ω	0.1981	0.1981	0.2054	0.1982	0.1978	←	←	←
	Λ	0.09919	0.09926	0.1006	0.09928	0.09889	←	←	←
0.40	Ω	0.4207	0.4178	0.4354	0.4189	0.4163	←	←	←
	Λ	0.2236	0.2213	0.2395	0.2217	0.2190	←	←	←
0.50	Ω	0.5460	0.5381	0.5622	0.5405	0.5361	0.5360	←	←
	Λ	0.3014	0.2942	0.3195	0.2953	0.2905	0.2904	←	←
0.80	Ω	0.9836	0.9387	0.9871	0.9492	0.9365	0.9360	←	←
	Λ	0.6113	0.5638	0.6156	0.5692	0.5541	0.5535	←	←
1.00	Ω	1.3207	1.2354	1.3045	1.2552	1.2337	1.2327	←	←
	Λ	0.8816	0.7857	0.8600	0.7963	0.7692	0.7679	0.7680	0.7681

CST : Classical shell theory.

FST: First order shear deformation shell theory (shear coefficient $\kappa^2 = 5/6$).
$M = 1$: Transverse normal strain $\gamma_{zz} = 0$.
$M = 1$ † : Transverse shear stress $s^{\alpha z} = 0$ (FST, $\kappa^2 = 1$).

When the natural frequency Ω_a goes to zero under the axial stress Λ, elastic buckling occurs and the critical buckling stress Λ_{cr} relates with the natural frequency Ω_0 as

$$\Lambda_{cr} = -\frac{4}{(1+\nu)s^2\pi^4}\left(\frac{H}{R}\right)^3\left(\frac{L}{H}\right)^4 \Omega_0{}^2. \tag{27}$$

The critical buckling stress of simply supported circular cylindrical shells subjected to axial compression can be predicted from the natural frequency of the shell without axial stress. The calculated critical buckling stresses corresponding to the lowest natural frequencies and vibration mode numbers are shown in Table 2. Two figures on the right shoulder of the natural frequencies and buckling stresses in Table 2 show that the first and second figures denote the wave numbers of *r* and *s*, respectively. These buckling stresses do not necessarily coincide with the lowest critical buckling stresses of the shells which may occur at different displacement mode numbers from the case of the lowest natural frequencies.

Table 2. Lowest natural frequency with vibration mode numbers and corresponding critical buckling stress (*M*=5)

		L/R						
H/R	Ω / Λ	2	3	4	6	8	10	20
0.05	Ω	0.02012^{31}	0.01316^{31}	0.01922^{21}	0.005731^{21}	0.004282^{21}	0.003743^{21}	0.001361^{11}
	Λ	0.004092^{31}	0.008862^{31}	0.01689^{21}	0.02689^{21}	0.04745^{21}	0.08851^{21}	0.01872^{11}
0.10	Ω	0.05639^{31}	0.03481^{21}	0.02404^{21}	0.01641^{21}	0.01432^{21}	0.01000^{11}	0.002725^{11}
	Λ	0.01607^{31}	0.03100^{21}	0.04673^{21}	0.1102^{21}	0.2653^{21}	0.3159^{11}	0.3753^{11}
0.20	Ω	0.1397^{21}	0.08807^{21}	0.06816^{21}	0.04781^{11}	0.02972^{11}	0.02009^{11}	0.005473^{11}
	Λ	0.04932^{21}	0.09923^{21}	0.1878^{21}	0.4679^{11}	0.5714^{11}	0.6375^{11}	0.7569^{11}
0.40	Ω	0.3686^{21}	0.2532^{21}	0.1782^{11}	0.09720^{11}	0.06038^{11}	0.04082^{11}	0.01112^{11}
	Λ	0.1717^{21}	0.4101^{21}	0.6420^{11}	0.9669^{11}	1.1792^{11}	1.3158^{11}	1.5624^{11}
0.50	Ω	0.5053^{21}	0.3316^{11}	0.2258^{11}	0.1229^{11}	0.07634^{11}	0.05161^{11}	0.01407^{11}
	Λ	0.2581^{21}	0.5627^{11}	0.8246^{11}	1.2367^{11}	1.5080^{11}	1.6827^{11}	2.0010^{11}
0.80	Ω	0.9360^{11}	0.5641^{11}	0.3807^{11}	0.2061^{11}	0.1279^{11}	0.08642^{11}	0.02357^{11}
	Λ	0.5535^{11}	1.0177^{11}	1.4650^{11}	2.1736^{11}	2.6456^{11}	2.9489^{11}	3.5097^{11}
1.00	Ω	1.2328^{11}	0.7375^{11}	0.4957^{11}	0.2677^{11}	0.1660^{11}	0.1122^{11}	0.03061^{11}
	Λ	0.7681^{11}	1.3916^{11}	1.9870^{11}	2.9337^{11}	3.5653^{11}	3.9765^{11}	4.7355^{11}

6. CONCLUSIONS

The following conclusions may be drawn from the present analysis :

(1) The natural frequencies and buckling stresses of thick circular cylindrical shells

calculated by using the classical thin shell theory are usually overpredicted. It has been pointed out that shear deformations and thickness changes have an important effect on the natural frequencies and buckling stresses of thick circular cylindrical shells.

(2) In order to verify the accuracy of the present results, the convergence properties of the numerical solutions according to the order of approximate theories have been examined. Without the assumption of $H/R \ll 1$, the present results obtained for $M = 5$ are considered to be accurate enough for very thick circular cylindrical shells. It may be noticed that the two-dimensional higher-order shell theory in thepresent paper can predict the natural frequencies and buckling stresses of a thick circular cylindrical shell.

(3) In the case of a simply supported circular cylindrical shell subjected to axial stress, the natural frequency can be expressed explicitly with reference to the natural frequency of a shell without axial stress. When the natural frequency goes to zero under axial compressions, elastic buckling occurs. The critical buckling stress can also be expressed in terms of the natural frequency of a shell without axial stress.

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