DISPLACEMENTS OF STRUCTURES
Application to classical structures

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1. Introduction

This study determines and compares the displacement at midspan of classical structures of different morphologies and deals with classical isostatic trusses and the simple beam.

The structure is made of a homogeneous material of allowable stress σ. It is completely isostatic, of height *H*, with two supports at a distance *L*, uniformly and vertically loaded by downward forces of a total intensity *F* (including its proper weight). For the trusses, the section of each beam between two nodes is considered as a constant.

Its dimensions are calculated considering the only criterion of resistance :

- For trusses : each bar is working at the allowable stress of the material.
- For the simple beam : the constant section is calculated so that the allowable stress is reached at midspan.

The phenomena of instability are not taken in account and the junctions between the members of the trusses are considered as perfect articulations. Stresses and strains resulting from shear are also not taken into account.

2. The indicator of displacement δ

An indicator of the vertical displacement of a structure at midspan, δ, independent of the characteristics of the material, the intensity of the forces *F* and the span *L* is first defined, and its value to different structures of various slenderness' *L/H* will be compared.

The indicator of displacement at midspan, dimensionless, can be described by : $\Delta = \dfrac{E\,\delta}{\sigma\,L}$

where :

- δ is the vertical displacement at midspan;
- E is the modulus of elasticity of the material;
- σ is the allowable stress of the material;
- L is the horizontal projection of the distance between the supports of the isostatic structure.

It corresponds to the vertical displacement at midspan of a unit structure (L = 1m), subjected to vertical loads of a total unit intensity (F = 1N), constituted of a material of a unit allowable stress (σ = 1Pa) and of a unit elasticity modulus (E =1Pa).

Indeed :

- In the case of a truss with "i" bars, taking the formula of O. Mohr (1874), and noting that each bar l_k of section Ω_k is subjected to the allowable stress σ, one obtains :

$$f_k = \sigma.\Omega_k \text{ , thus } \delta = \sum_{k=1}^{k=i} \frac{f_k \, f_k^1}{E\Omega_k} l_k = \frac{\sigma L}{E} \sum_{k=1}^{k=i} f_k^1 \, \frac{l_k}{L}$$

Where :
- f_k^1 is the axial stress in bar l_k, when the structure is subjected to a unit vertical load at midspan.
- f_k is the axial stress in the same bar l_k, when the structure is subjected to uniformly distributed vertical and downward forces of total intensity F.

Thus : $\Delta = \frac{E\delta}{\sigma L} = \sum_{k=1}^{k=i} f_k^1 \quad \frac{l_k}{L}$

- In the case of a straight or curved bar subjected to axial load, with a uniform and continuous section subjected at one of its points to the allowable stress or with continuous but variable section subjected at any point to the same allowable stress $\sigma = F / \Omega$, one obtains :

$$\delta \quad \frac{FL}{E\Omega} \; ; \; \delta \quad \frac{L\sigma}{E} \quad \text{thus} \; : \Delta = \frac{E\delta}{L\sigma} = \text{constant}$$

- In the case of a beam in bending, with a constant and continuous section and symmetrical with respect to the neutral axis, subjected on one of its points, at midspan, to the allowable stress σ, one obtains :

$$\sigma \quad \frac{FLH}{I}$$

Furthermore, as $\delta \div \frac{FL^3}{E\,I}$ and $\delta \div \frac{\sigma\, L}{E} \frac{L}{H}$, thus $\Delta = \frac{E\delta}{\sigma L} \quad \frac{L}{H}$

3. The indicator of displacement for the WARREN truss

Figure 1 illustrates the stresses in the bars of a WARREN truss, with "n" even, subjected firstly to "n " vertical loads F/n of total intensity F and secondly to one single unit load at midspan.

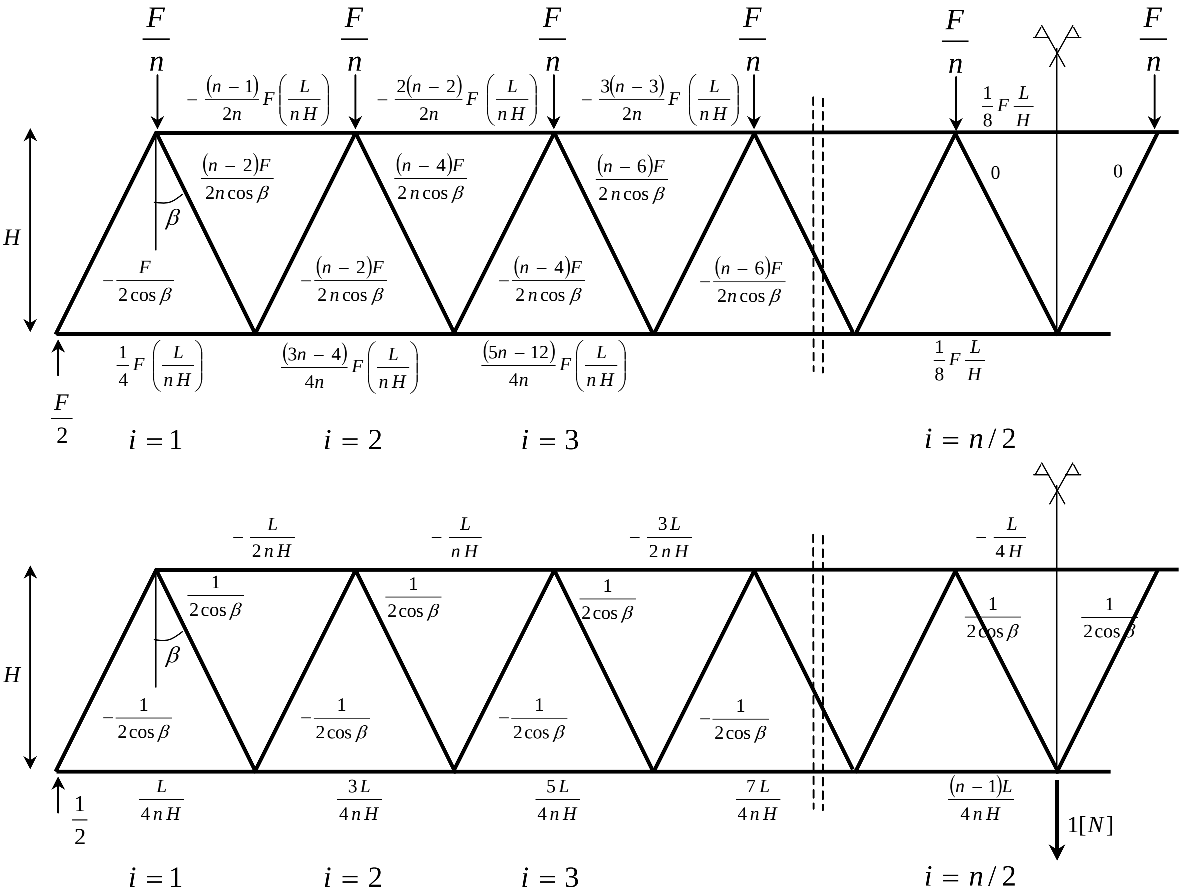

Figure 1

The indicator of displacement is composed of 3 parts, corresponding respectively to the lower chord, the upper chord and the diagonals. This last part is obtained noting that the stress corresponding to the two central diagonals in the main structure subjected to "n " vertical loads of total intensity "F" is equal to zero .

$$\Delta = \sum f_k^1 \frac{l_k}{L} = 2\sum_{i=1}^{i=n/2}\left[\left((2i-1)\frac{L}{4nH}\right)\frac{1}{n}\right] + 2\left[\frac{1}{2}\frac{L}{4H} + \sum_{i=1}^{i=n/2-1}\left(\frac{iL}{2nH}\right)\right]\frac{1}{n} + (n-1)\frac{1}{\cos\beta}\frac{H}{\cos\beta\, L}$$

After simplification of the expression, one finds : $\Delta_{W,n\,even} = (n-1)\frac{H}{L} + \frac{n^2+n-1}{4n^2}\frac{L}{H}$

For an uneven number "n" of panels, the same reasoning as above leads to the following result :

$$\Delta_{W,n\,uneven} = (n-1)\frac{H}{L} + \frac{(n^2+n-2)}{4n^2}\frac{L}{H}$$

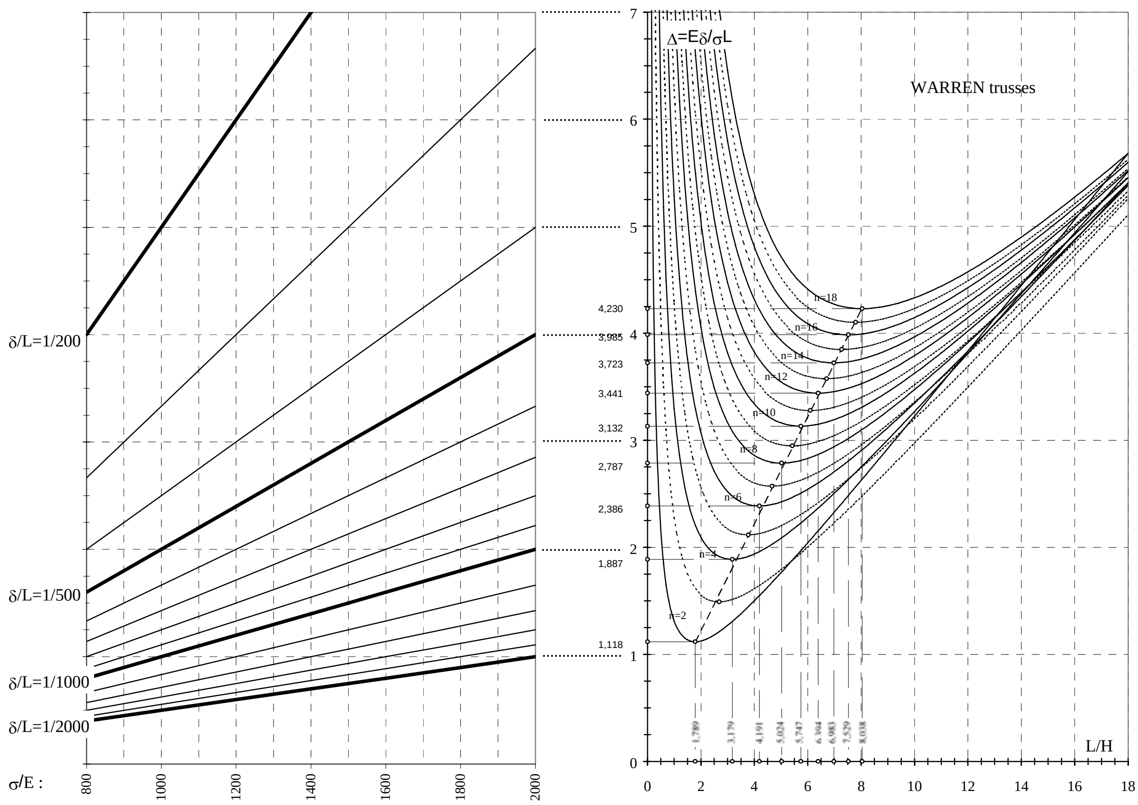

Figure 3

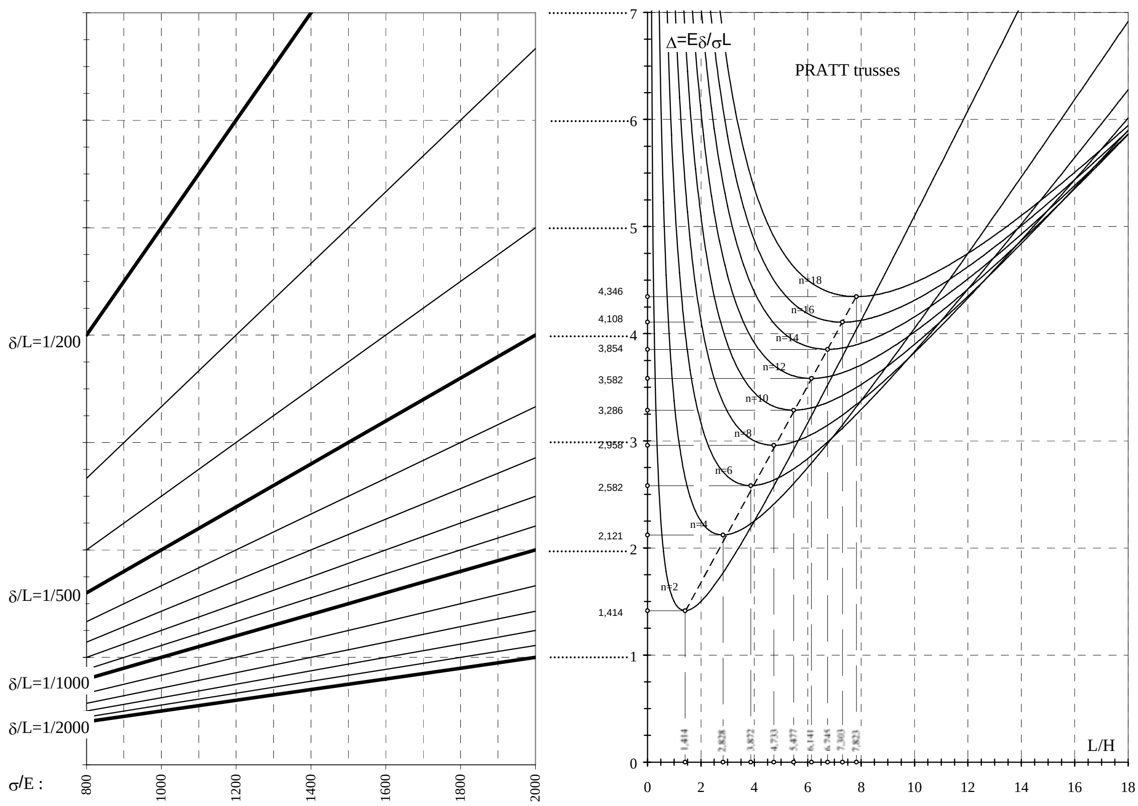

Figure 4

The right of figure 3 illustrates the values of Δ for the WARREN truss for a number "*n*" of panels between 2 and 18, and for a slenderness *L/H* between 0 and 18. One notes that :

- The optimum optimorum is obtained for n = 2 and corresponds to L/H = 1,789 and Δ = 1,118;
- The envelope curve of the WARREN truss is obtained with the curves corresponding to n=2 and n=3;
- The minima of the curves for n=2, 4, …16 and for n=1, 3 ..17 are almost located on a straight line.

The left part of figure 3 has to be considered in conjunction with the right part of figure 3. It gives the values of δ/L for various E/σ. It allows immediate reading of the relative deflection for the structures in any material, such as concrete (E/σ≈2000), mild steel (E/σ≈1500) or wood (E/σ≈1000)

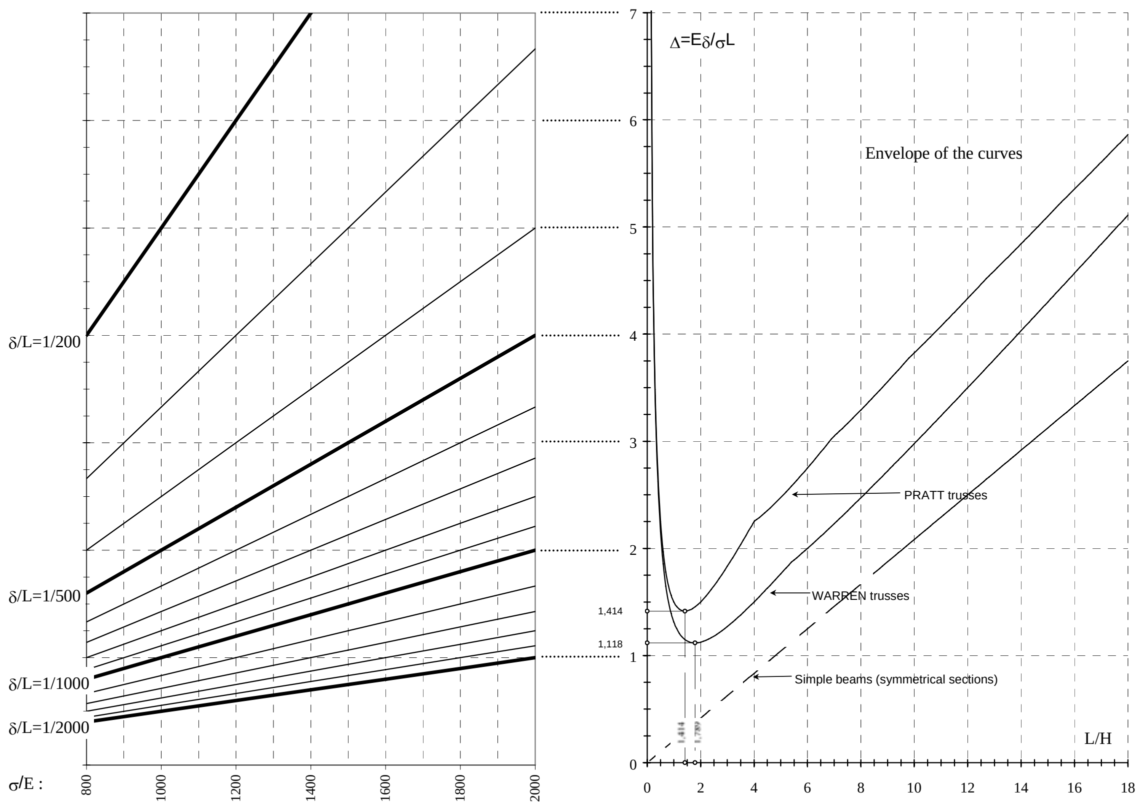

Figure 5

4. Indicator of displacement for the PRATT truss

Figure 6 illustrates the stresses in the bars of a PRATT truss subjected to one single unit load at midspan. The same reasoning as above leads to the following result :

$$\Delta_P = (n-1)\frac{H}{L} + \frac{(n+2)}{4n}\frac{L}{H}$$

Figure 4 illustrates the values of Δ for the PRATT truss for a number "*n*" of panels between 2 and 18, and for a slenderness *L/H* between 0 and 18. One notes that :

- The optimum optimorum for the PRATT truss is obtained for $L/H = \Delta = \sqrt{2}$;
- The minima of the curves for n=2, 4, …16 and for n=1, 3 ..17 are almost located on a straight line.

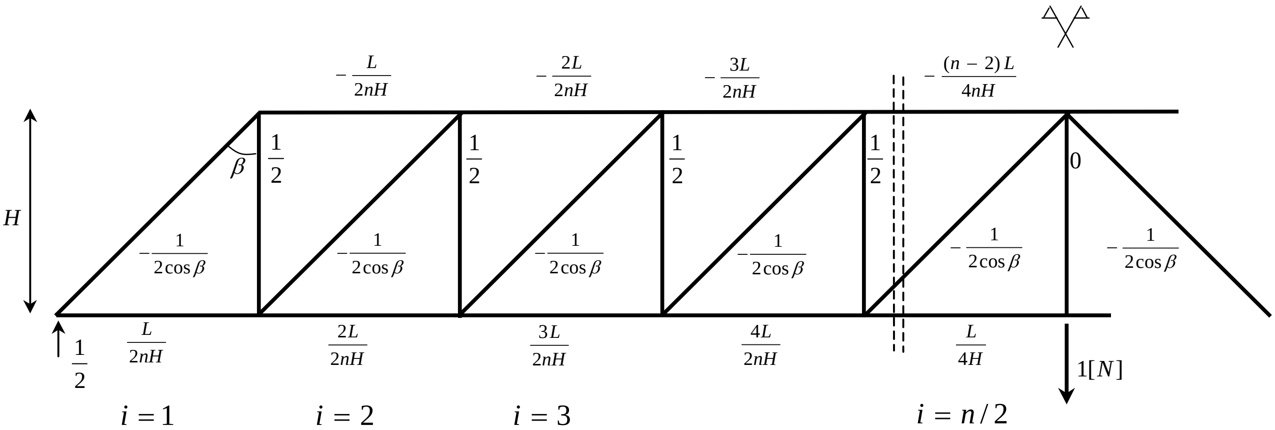

Figure 6

5. Indicator of displacement for the single beam

Noting that the displacement at midspan of a single beam subjected to a uniformly distributed load *P* [N/m] is $\delta = \frac{5PL^4}{384EI}$ and that the bending moment at midspan is $\frac{PL^2}{8}$, so that the allowable stress at that midsection is $\sigma = \frac{PL^2H}{16I}$, one finds : $\delta = \frac{5\sigma L^2}{24EH}$.

The displacement due to bending at midspan of a unit structure ($L = 1$ m), subjected to vertical loads of a total unit intensity ($F = 1$N), constituted of a material of a unit allowable stress resistance ($\sigma = 1$ Pa) and of a unit elasticity modulus ($E = 1$ Pa) is thus equal to $\Delta = \frac{5}{24}\frac{L}{H}$. This relationship is not valid for low *L/H* ratios when the shear deformation are to be taken in account.

The indicator of displacement corresponding to a single beam is thus linearly proportional to the slenderness *L/H*. Figure 5 illustrates the values of Δ for the simple beam, with respect to the envelope curves of Δ corresponding to the WARREN and PRATT trusses.

6. Conclusion

Figure 5 suggests that, for any slenderness, the WARREN truss is always less deformable than the PRATT truss, and the simple bended beam is always less deformable than both trusses.
Further studies will search for the values of Δ of other structures, such as arches or cables.