VOLUME OF STRUCTURES
Application to classical and harmonic structures

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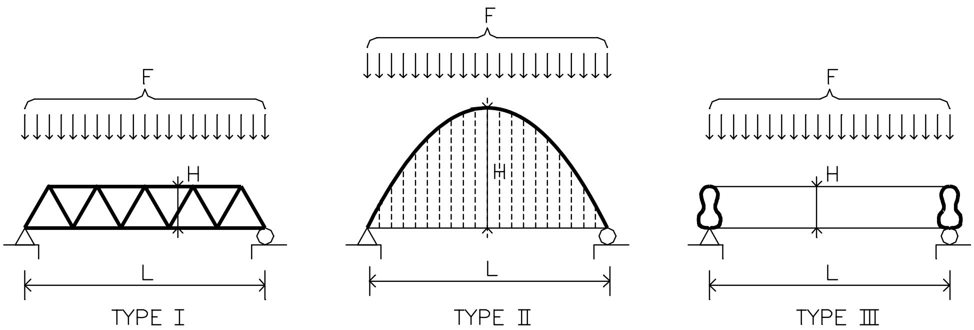
1. Introduction

Several parameters influence the optimization of a structure under a given principal load case. Resistance and stiffness of a construction depend most certainly on material, but also, on its geometry. Moreover, for each load case there is a corresponding different optimal geometry. Its cost is mostly determined by local traditions and socio-economic characteristics of the location.
Finally, physical performances of a construction as a whole, as fire resistance, and the physiological needs that it must satisfy, influence decisively the structural choice.
In spite of these considerations and at a philosophical level, it remains useful to think about structural morphology of simple building systems under uniform vertical loads, and thus to search to limit the volume of materials used, all other criteria remaining constant.
The study concerns the isostatic truss beam, the arch and the simple beam, composed of hollow or full symmetrical and constant section.



The structure is made of a homogeneous material of allowable stress σ. It is completely isostatic, of height *H*, with two supports distant of a length *L*, uniformly and vertically loaded by downward forces of total intensity *F* (including its proper weight). Furthermore, the section of each beam between two nodes is considered as a constant. Its dimensions are calculated considering the only criterion of resistance :

- For trusses : each beam is working at the allowable stress of the material.
- For the arch : the section is adapted along the middle fibre to fit to the allowable stress of the material.

- For the simple beam : the section, constant, is calculated so that the allowable tensile stress in flexion is reached at midspan or the allowable shearstress at supports.

The phenomena of instability are not taken in account and the junctions between the members of the trusses are considered as perfect articulations. This study determines and compares the volume of material required for these different morphologies.
An indicator of the volume of any structure, *W*, independent of the characteristics of the material, the intensity of the forces *F* and the span *L,* is first defined, and its value will be compared to different structures of various slendernesses *L/H*.

1. The indicator of volume W

The indicator of volume, without any dimensions, can be defined by $W = \frac{\sigma V}{FL}$, where :

- V is the least total volume of the structure;
- σ is the allowable stress of the material wich composes the structure;
- F is the sum of the intensity of the loads applied to the structure;
- L is the horizontal projection of the distance between the supports of the isostatic structure.

W is related to the volume of a structure of a unit span (*L*= 1m) subjected to forces of a total unit intensity (*F*= 1N), composed of a material of a unit allowable stress (σ =1 Pa).

2. Study of trusses

2.1. Classical trusses

The study concerns only the main morphologies of trusses :

- The truss made up of members in rectangular panels stiffened by diagonals and known as the PRATT beam, patended in 1842 in the USA;
- The truss composed of equilateral triangular panels, first built in 1845 and in Belgium by NEVILLE, known as the WARREN beam, patented in 1848 in England.

Figures 2 and 3 represent *W* in function of the slenderness *L/H*, varying from 0.1 to 18 for PRATT and WARREN beams, respectively for a number of meshes between 4 and 18 by step 2, and "*n*" [1] between 2 and 18. Those figures lead to the following comments :

- The envelope curve, $W_{\min}$, is always composed of parts of the beam's work curves W_n with an even number of panels (n = 2, 4, 6, 8, …);
- These curves can be approximated by straight parallel lines for L/H≥6;
  $W_{\min,PRATT}$ = 0,780 + 0,164 L / H et $W_{\min,WARREN}$ = 0,561 + 0,164 L / H
- The optimal PRATT beams are thus always more voluminous than the corresponding WARREN (14% to 6% for a slenderness between 6 and 18).
- The observation of the points on each W_n curve corresponding to various angles α at the apex of the panels shows that the optimum angle varies approximately between 30° and

[1] As a convention, a WARREN truss with "*n*" panels contains "2n-1" triangles and a PRATT truss "2n-2" triangles; "*n*" being greater than 1.

45° for the PRATT and between 60° and 90° for the WARREN beam. These points are located approximatively on straight lines parallel to the lines described by the above equations.

One shows that the value of W_n , for WARREN beams on figure 3 is equal to :

for " n " even : $W_{np} = \dfrac{n}{2}\dfrac{H}{L} + \left(\dfrac{4n^2 + 3n + 2}{24n^2}\right)\dfrac{L}{H}$

and for " n " uneven : $W_{ni} = \left(\dfrac{n^2+1}{2n}\right)\dfrac{H}{L} + \left(\dfrac{4n^3 + 3n^2 + 2n + 3}{24n^3}\right)\dfrac{L}{H}$

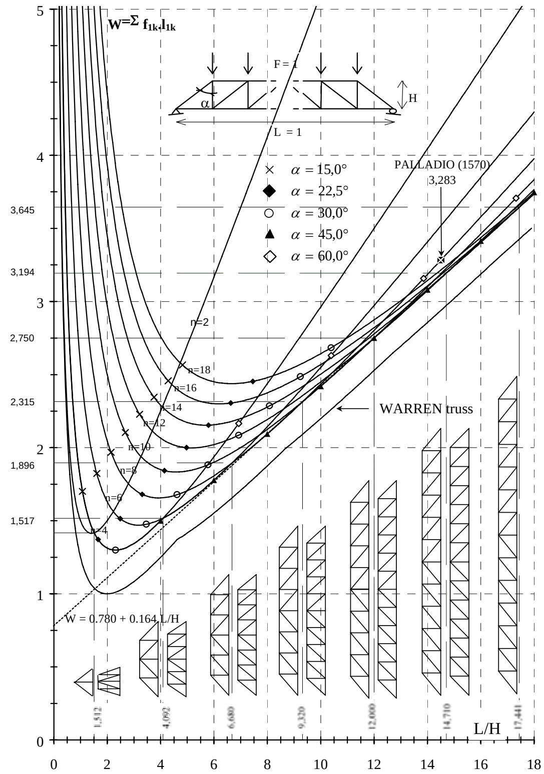


Figure 2

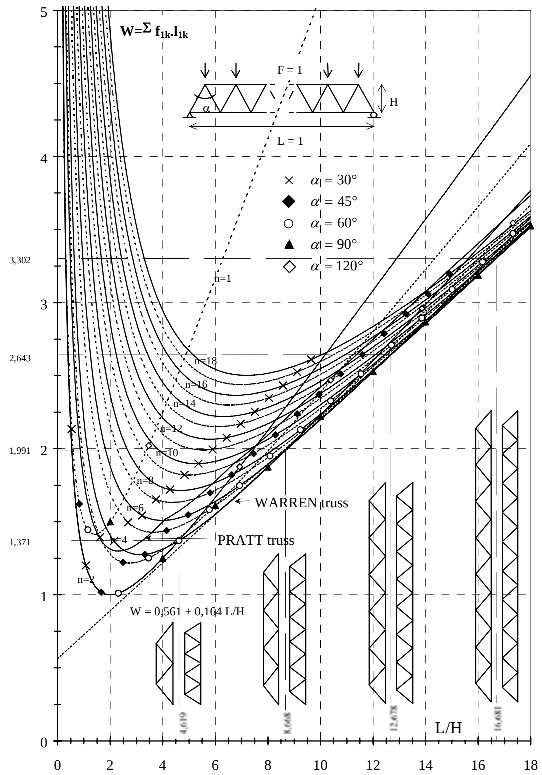


Figure 3

2.2. HARMONIC trusses.

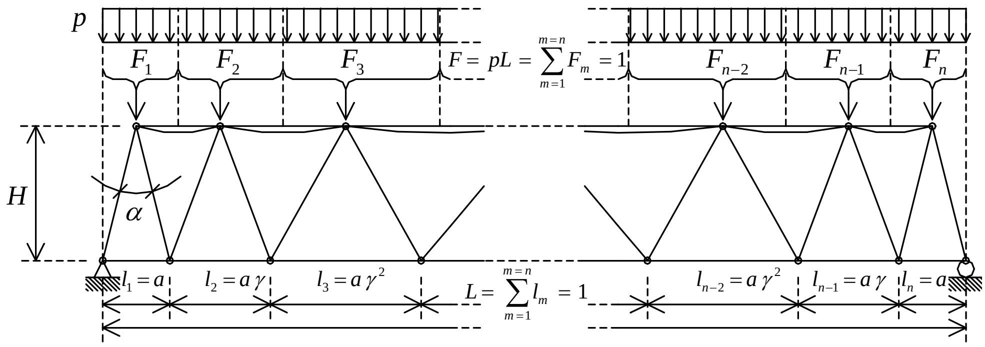


Figure 4

The harmonic truss, part of the family of harmonic structures develloped by Samyn&Partners, can be described as a truss whose panels are of variable dimensions, following a regular

progression. Figure 4 shows, for example, a WARREN beam with isosceles panels whose dimensions follow a geometrical progression caracterized by the parameters γ and a.
It has been shown [2] that this type of truss can be, in a certain number of cases, even lighter than the corresponding WARREN truss (each load is applied to a node of the structure). If the external loads are not directly applied to the nodes, it is interesting to determine the aditionnal quantity of material needed in the *"n"* members l_m of the lower chord (of length L=1 m) uniformly loaded (by p=F/L=1 Mpa) to transmit loads to the nodes. One must take into account their variation of length related to a geometrical progression of parameter γ (the lower chord is made up of "*n*" members of same length when γ=1).
It as been shown [2] that for a given length L, a series of isostatic beams l_m of constant span (γ=1) is always less voluminous than a series of isostatic beams of variable sections. However, this disadvantage can be solved by using MULTI-TRUSS beams.

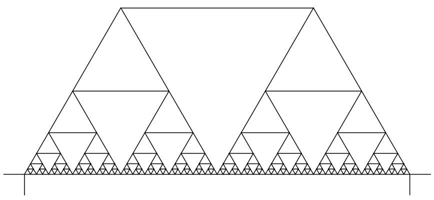
2.3. The MULTI-TRUSS beam.

The MULTI-TRUSS beam is first composed out of a classical truss beam. The lower or upper chord of each of its "*n* " panels is then constitued with a first row of truss beams "*n*" times smaller. The "n^2 " segments of that horizontal chord are in turn composed of a second row of truss beams and so on. Assuming that :

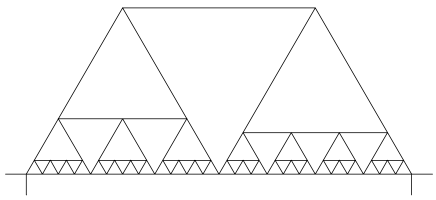
- $W_{n,j}$ is the volume of the MULTI-TRUSS beam with "*n*" panels, each of these panels being divided "*j* " times, or from a row "*j* ";
- $W_{n,0}$ or W_n is the volume of the initial truss beam
- $W_{n,\infty}$ represents then the volume of the MULTI-TRUSS which is divided an infinite number of times, and capable of transmitting to the supports, the uniformly distributed load only with compressed or tight members, barring any member in flexion.

As it can be seen on figure 5 below, the secondary truss beam of each row, replacing successively the "*n* " parts of the lower chords, may have the same form has the initial truss (One refers to it as a HOMOGENEOUS MULTI-TRUSS or simply as a MULTI-TRUSS). It can also be similar but either with a different number of panels (VARIABLE MULTI-TRUSS), of proportions (DEFORMED MULTI-TRUSS) or even be of a different type (MIXED MULTI-TRUSS).

Figure 5



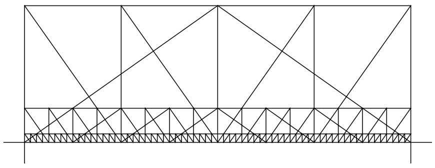
HOMOGENOUS MULTI-TRUSS



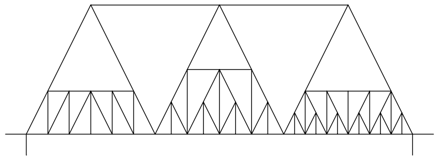
VARIABLE MULTI-TRUSS

[2] - Proceedings of the ASIA-PACIFIC CONFERENCE ON SHELL AND SPATIAL STRUCTURES. CCES-IASS APCS '96-May 21-25, 1996 Beijing-China. Ph. Samyn and L. Kaisin : "Harmonic Structures : The Case of an isostatic Truss Beam"; pp. 255-262.
- Lionel Huftier, Faculté Polytechnique de Mons, final thesis, 95-96, 49 pp. and annexes; "Studies about harmonic trusses and comparison with the Warren truss"

Figure 5



DEFORMED MULTI-TRUSS



MIXED MULTI-TRUSS

Figure 6 shows a few MULTI-TRUSSES resulting from a WARREN beam, referred to as a MULTI-WARREN and of volume $WW_{n,j}$. Figure 7 shows a few MULTI-TRUSSES ensuing from a PRATT truss, called MULTI-PRATT, and of volume $WP_{n,j}$. Similar figures can be established for other type of trusses, including for harmonic trusses, called MULTIHARMONICS, and of volume $WH_{n,j}$.

For a HOMOGENOUS MULTI-TRUSS, it can be proved that, on one hand :

$$W_{n,j} = W_{n,0} \sum_{k=0}^{k=j} \frac{1}{n^k} \quad \text{, as} \quad \lim_{u\to\infty} \sum_{k=0}^{k=u} \frac{1}{n^k} = \frac{n}{n-1} \quad , W_{n,\infty} = W_{n,0} \ \frac{n}{n-1}$$

and, one the other hand, that : $\dfrac{W_{n,\infty}}{W_{n,0}} = \dfrac{WW_{n,\infty}}{WW_{n,0}} = \dfrac{WP_{n,\infty}}{WP_{n,0}}$, and : $\dfrac{W_{n,j}}{W_{n,0}} = \dfrac{WW_{n,j}}{WW_{n,0}} = \dfrac{WP_{n,j}}{WP_{n,0}}$

$WW_{2,5}$ $WW_{3,3}$ $WW_{4,2}$ $WW_{5,2}$ $WW_{6,2}$

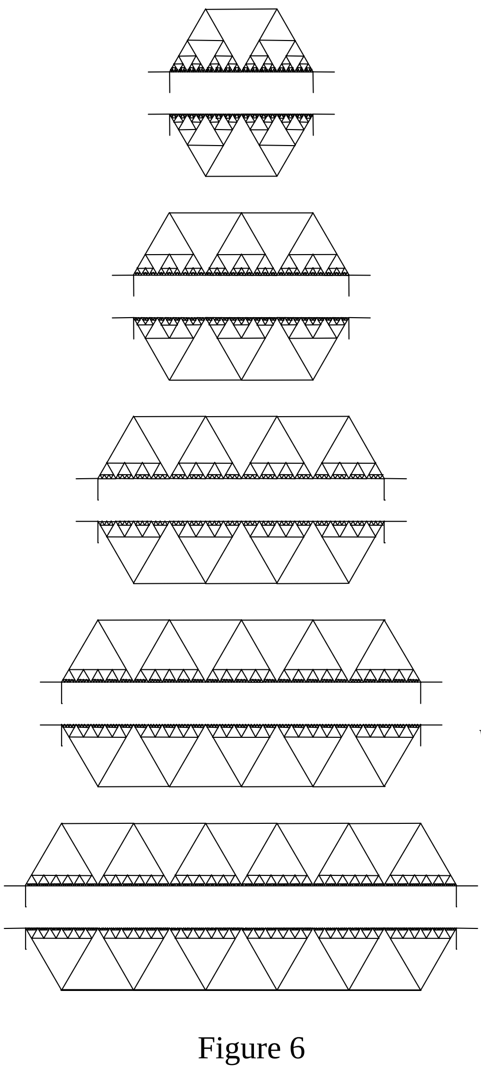


Figure 6

$WP_{2,5}$ $WP_{3,3}$ $WP_{4,2}$ $WP_{5,2}$ $WP_{6,2}$

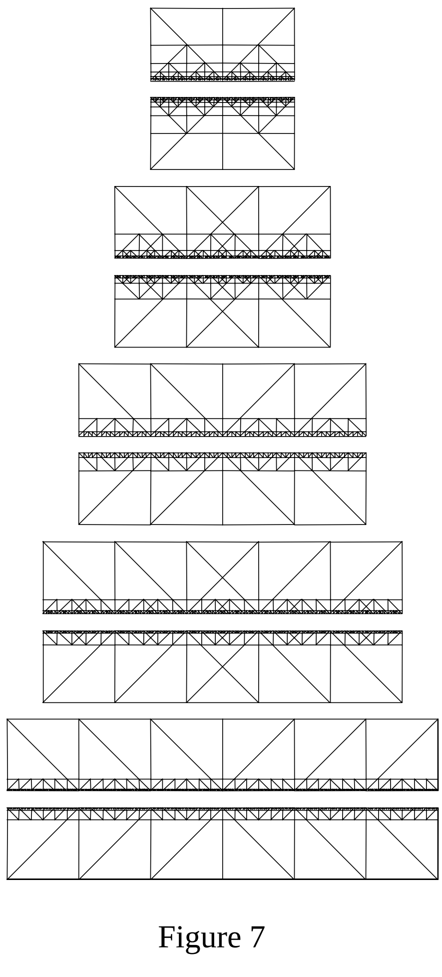


Figure 7

The width and the section of each of the corresponding "*k*" members of a truss with "*n*" panels at each intermediate row "*r* ", provided the b_k/h_k proportions are constant, are thus, :

$$b_{k,n,r} = \frac{b_{k,n,0}}{\sqrt{n^r}} \; ; \quad \Omega_{k,n,r} = \frac{\Omega_{k,n,0}}{n^r}$$

The section or width, resulting from an element at a given point of a MULTI-TRUSS at row "*j*" is the sum of the sections $\Omega_{k,n,r}$, or of the width $b_{k,n,r}$, of the members belonging to the intermediate rows "*r*" from 1 to "*j*" passing through this point. Figure 8 below shows a MULTI-WARREN with 2 panels with 4 rows, where the thickness of the line is adjusted to the dimension of the section of the members (on the left) or to their width (on the right).

Figure 8 :

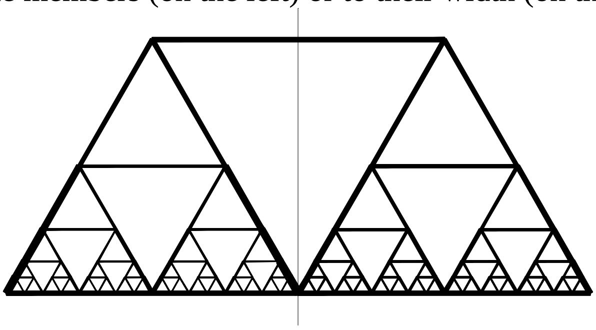


Figure 9 shows the rapid convergence of the serie $\sum_{k=0}^{k=j} \frac{1}{n^k}$, so that the volume of a two-row MULTI-TRUSS composed of more than 7 panels is already almost equal to the volume of the same beam with an infinite number of rows, just as the one of a single row MULTI-TRUSS with more than 18 panels.

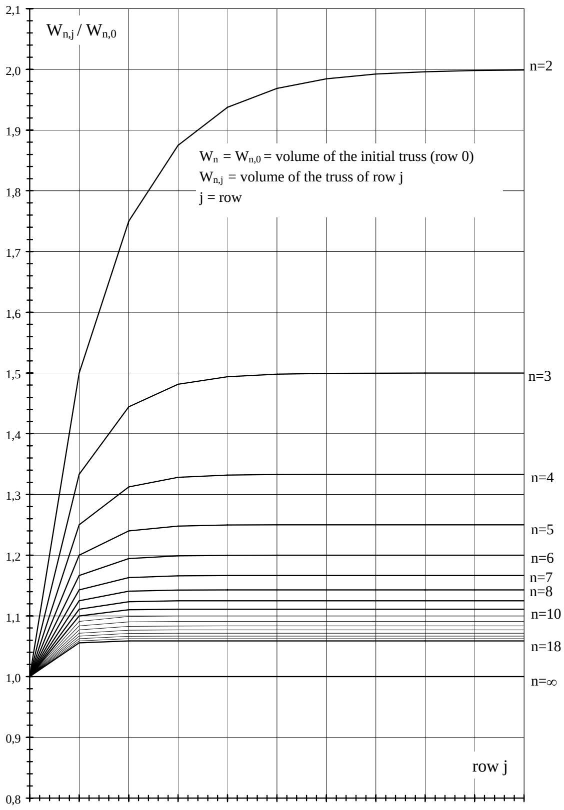


Figure 9

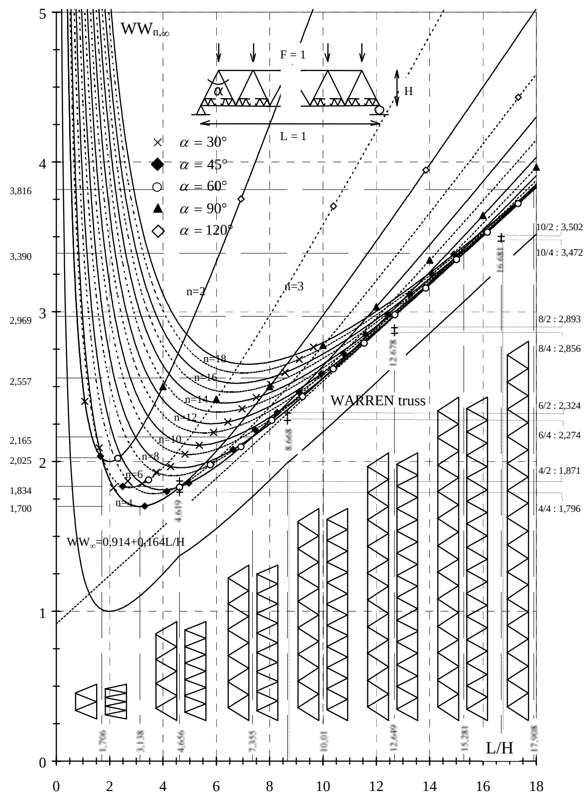


Figure 10

2.4. The HOMOGENOUS MULTI-WARREN beam.

Figure 10 gives values of volume $WW_{n,\infty}$ of MULTI-WARREN beams in function of a slenderness L/H ratio contained between 0.1 and 18, and for "*n*" included between 2 and 18. It is to be related to figure 3. It suggests the following comments :

- One notices once more that the envelope curve $WW_{n,\infty}$, always composed of segments of the curves belonging to beams with an even number of panels, can be approximated by a straight line which is parallel to the line related to a simple PRATT or WARREN beam : $WW_{\infty\,\min} = 0{,}914 + 0{,}164\ L/H$
- The minimum minimorum, related to the simple WARREN truss with an L/H=2 and two panels, is obtained for a MULTI-WARREN with an L/H=3,138 and four panels.
- From the above relations, one shows that the $WW_{n,\infty}$ of the MULTI-WARREN truss of figure 10, with an even number of panels, is equal to $WW_{n,\infty,p} = \frac{n^2}{2(n-1)}\frac{H}{L} + \frac{4n^2+3n+2}{24n(n-1)}\frac{L}{H}$
- The points on each curve $WW_{n,\infty}$ related to different angle openings show that the optimum angle varies between 40° and 65° , and stabilizes at, approximatively, 61° for a great number of panels, against 86° for the simple WARREN truss. Hence, these points are, once more, approximatively situated on straight lines, parallel to the lines described above.
- For a given slenderness L/H, the optimum for a MULTI-WARREN always shows a greater number of panels "*n*" than the optimum WARREN truss (n = 6 instead of 4 for an $L/H \approx 5$; 8 instead of 6 for an $L/H \approx 9$; 12 instead of 8 for an $L/H \approx 13$; and 14 instead of 10 for an $L/H \approx 17$).

3. Study of the parabolic arch

The parabolic doubled pinned arch of a unit span (L=1 m) with a constant axial unit stress ($\sigma = 1\text{Pa}$) at each point subjected to a uniformly distributed horizontal load and of a total unit intensity ($F = 1\text{N}$) presents a volume $W = \frac{2}{3}\ H/L + \frac{1}{8}\ L/H$, with a minimum $W = 0{,}577$ for an $L/H = 2{,}309$.
The same arch, equipped with a tie rod joining the extremities of the arch in order to take up the inward thrust, and whereby one of the supports is pinned and the other one free to move lateraly, has a volume $W = \frac{2}{3}\ H/L + \frac{1}{4}\ L/H$, with a minimum $W = 0{,}816$ for an $L/H = 1.633$.
The same arch, either equipped or not with a tie bar, and with an infinite number of hangers, bellow the arch, has a volume :

- Without any tie bar :
  $W = \frac{4}{3}\frac{H}{L} + \frac{1}{8}\frac{L}{H}$; with a minimum $W = 0{,}816$ for an $L/H = 3{,}266$.
- With tie bar :
  $W = \frac{4}{3}\frac{H}{L} + \frac{1}{4}\frac{L}{H}$; with a minimum $W = 1{,}155$ for an $L/H = 2{,}309$.

The same arch, either equipped or not with a tie bar, and with an infinite number of compressive rods, above the arch, has a volume :

- Without any tie bar :
  $W = H/L + 1/8H$; with a minimum $W = 0{,}707$ for an $L/H = 2.828$.
- With tie bar :
  $W = H/L + 1/4H$; with a minimum $W = 1$, for an $L/H = 2$. It is to be noted that this volume indicator is equal to W_2 related to a WARREN truss with 2 panels.

Figure 11 shows the optimal proportions for these various cases. Figure 12 shows *W* for these cases in function of L/H, with regard to the value of *W* for the envelope curves of WARREN and MULTI-WARREN trusses as well as for the simple beam, detailled in the next point.

Figure 11 :

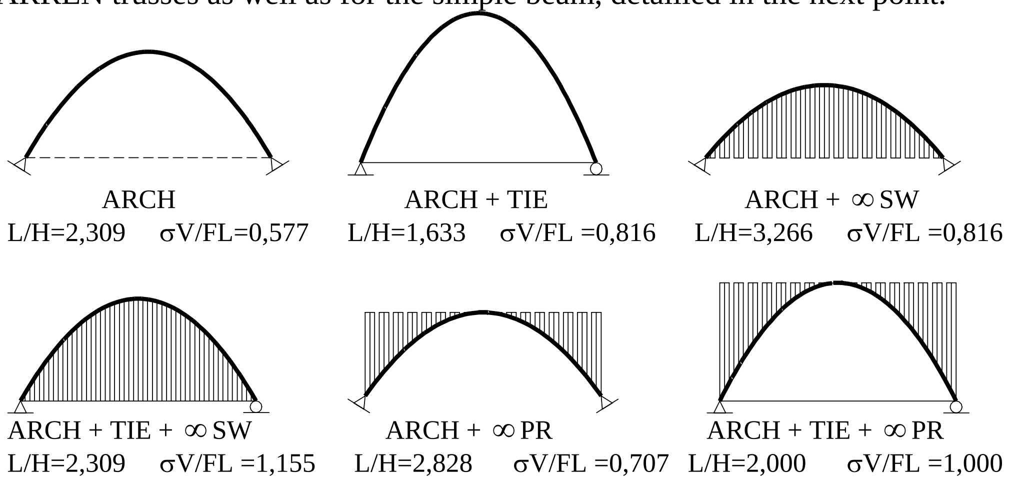
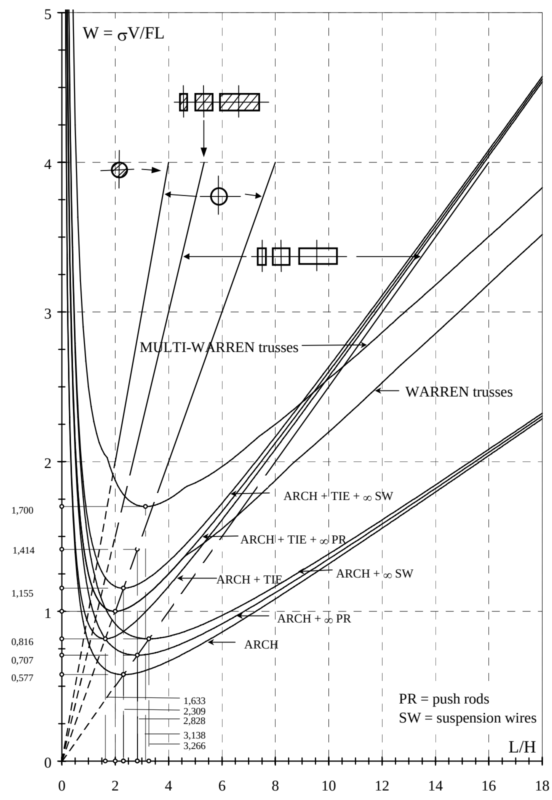


Figure 12 :



4. Study of the simple beam

The simple beam, with a symetrical constant section, of a height *H*, of an inertia *I*, of a unit span (L=1), with a unit stress ($\sigma = 1$ Pa) which is reached at midspan, subjected to a constant and uniformly distributed vertical load of a total unit intensity ($F = 1$N), has a volume $W = \dfrac{\Omega H^2}{16I}\ L/H$.

- For a full rectangular section and a full circular section, it becomes, respectively, $W = \dfrac{3}{4}\ L/H$ and $W = L/H$.
- For a hollow rectangular section, of thickness *e* which is lower or equal to the lowest half of its width *B* or to its height *H* , the indicator of volume corresponds to :
$$W = \frac{3}{2}\ \frac{(e/B)(H/B)^2(1+H/B-2e/B)}{(H/B)^3+(2e/B-1)(H/B-2e/B)^2}\ L/H$$
- For a hollow circular section, of thickness *e* which is lower or equal to half its diameter, one obtains $W = \dfrac{1}{2-4(e/H)+4(e/H)^2}\ L/H$

For full sections, *W* only depends on *L/H*. For tubes, it also depends on the ratio *e/H* or *e/B*. For rectangular hollow sections, it also depends on *H/B*. The values of $\dfrac{\Omega H^2}{16I}$ are illustrated on figures 13 and 14.

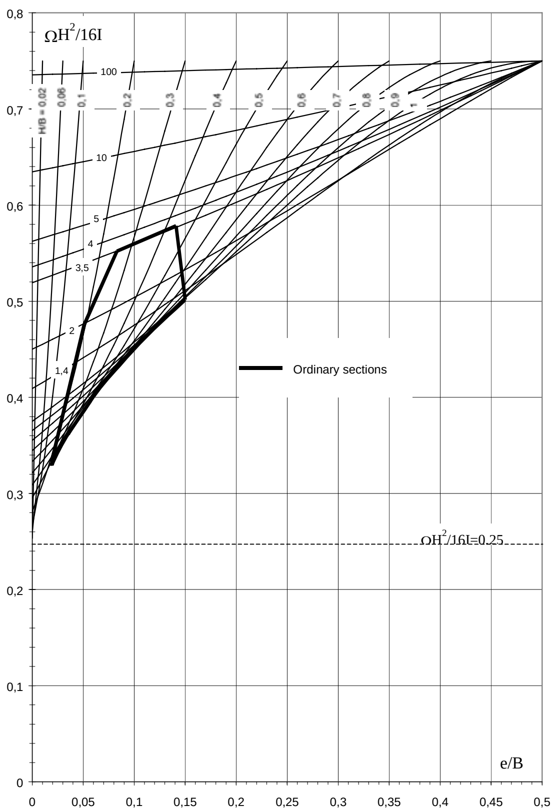


Figure 13

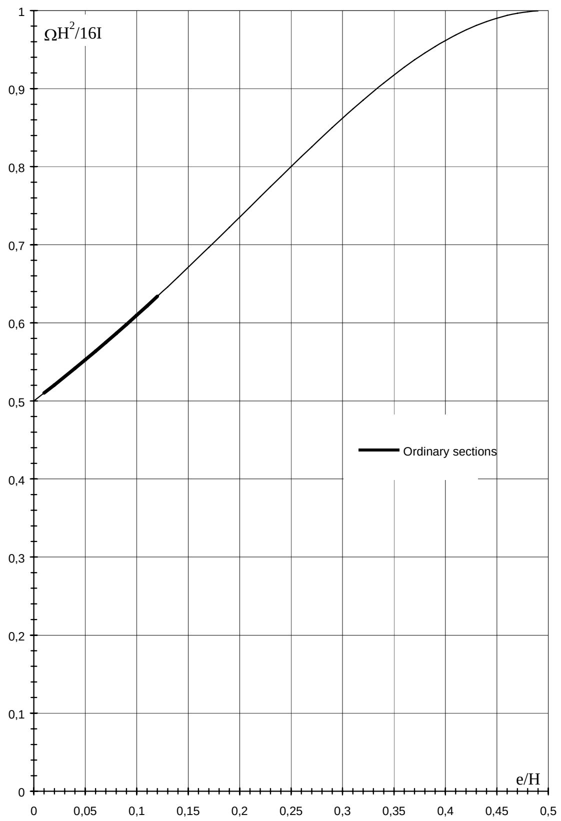


Figure 14

Those figures suggest that :

For rectangular hollow sections :

- The optimal value of W is obtained for a nul thickness, regardless the value of H/B;
- The optimum optimorum corresponds to $W=0,25\ L/H$ for a hollow section of infinitesimal thickness and infinitely larger than high;
- For a given thickness (different from zero), there always exists an optimal value of the ratio H/B, and for values of e/B lower than 0,15 (that means for the majority of hollow sections), the optimum corresponds to a ratio H/B lower than 1;
- The influence of the thickness on W decreases as H/B increases.

For circular hollow sections :

- The value of W decreases if the thickness of the tube decreases;
- The optimum optimorum corresponds to $W=0,5\ L/H$ and to an infinitesimal thickness.

Figure 12 shows W for these cases in function of L/H, with regard to the value of W for the WARREN and MULTI-WARREN trusses (envelope curves), as well as for the arch.

The lowest W value are though limited by the allowable shear stress at supports, which occurs for $W = \sqrt{3}$ in the case of a round tube.

5. Conclusions

One has thus shown that, for the classical analysed structures, the range of W is quite limited and between 0,577 and 4, for a slenderness L/H between 1 and 16. These graphs allow furthermore for a handy and fast comparison of structural systems at the preliminary design stage.